

National University of Singapore

MA2002 - Calculus

Midterm Exam

(AY2026/2027 Semester 1)

Time allowed: 1 hour

INSTRUCTIONS TO STUDENTS

1. This test will be proctored using Exemplify.
2. This is a NON-SECURE BLOCK INTERNET assessment. You are allowed to refer to any soft copy or hard copy notes. All soft copy notes must be accessed through the device you are using for the exam.
3. This test consists of **12** multiple response questions, each carrying **1** mark.
4. You are required to answer **ALL** questions. The examinable scope is Sections 1.1 to 5.6.
5. You are not told how many options are correct in any question, and there is no limit on how many you may select.
6. Marks are awarded for each correct option you select and deducted for each incorrect option you select. The lowest score for any question is 0. It is therefore to your advantage to select an option whenever you judge it more likely to be true than false, and to leave it unselected otherwise.
7. Submit your answers in Exemplify.

1. Let g be an odd function defined on $\mathbb{R}$, and let h be a function such that

$$0 \leq g(x) \leq h(x) \text{ for all } x \in (0, 1), \quad \lim_{x \rightarrow 0^+} h(x) = 0.$$

Which of the following statements must be true? Select all that apply.

- (a) $\lim_{x \rightarrow 0} g(x) = 0$.
- (b) $g(x) \leq 0$ for all $x \in (-1, 0)$.
- (c) $g(0) = 0$.
- (d) $\lim_{x \rightarrow 0^-} h(x) = 0$.
- (e) g is differentiable at 0.
- (f) $\lim_{x \rightarrow 0^+} \frac{g(x)}{h(x)}$ exists.

Solution:

- (a) The Squeeze Theorem applies only on $(0, 1)$ and gives $\lim_{x \rightarrow 0^+} g(x) = 0$. Parity completes it: for $x \in (-1, 0)$, $g(x) = -g(-x)$ and $-h(-x) \leq g(x) \leq 0$, so $\lim_{x \rightarrow 0^-} g(x) = 0$ as well.
 - (b) Since $g \geq 0$ on $(0, 1)$ and g is odd, $g(x) = -g(-x) \leq 0$ on $(-1, 0)$. Parity transfers the sign, not only the limit.
 - (c) g is odd and defined at 0, so $g(0) = g(-0) = -g(0)$, giving $2g(0) = 0$.
 - (d) False. Nothing is assumed about h outside $(0, 1)$. Take $g \equiv 0$, $h(x) = x$ on $(0, 1)$ and $h \equiv 5$ elsewhere.
 - (e) False. Take $h(x) = \sqrt{x}$ and $g(x) = \sqrt{x}$ on $(0, 1)$, extended oddly with $g(0) = 0$. Then $g(x)/x = 1/\sqrt{x} \rightarrow \infty$.
 - (f) False. Take $h(x) = x$ and $g(x) = x \sin^2(1/x)$ on $(0, 1)$, extended oddly. The bound holds, but $g(x)/h(x) = \sin^2(1/x)$ has no limit. The Squeeze Theorem pins down the limit, never the relative size.
2. Suppose that $(f(x))^2$ is differentiable at $x = a$. Which of the following statements must be true? Select all that apply.
- (a) $(f(x))^{2026}$ is differentiable at a .
 - (b) $|f(x)|^3$ is differentiable at a .
 - (c) $(f(x))^2$ is continuous at a .
 - (d) $|f(x)|$ is differentiable at a .
 - (e) f is continuous at a .

Question 2 continues...

(f) $(f(x))^2$ is differentiable on some open interval containing a .

Solution: Write $g = f^2$, so g is differentiable at a and $g \geq 0$. Only expressions depending on g alone are protected; f , f^3 and $f|f|$ all require the sign of f , which g does not record.

(a) $f^{2026} = g^{1013}$, a product of functions differentiable at a .

(b) $|f|^3 = g^{3/2}$. If $g(a) > 0$, the chain rule applies. If $g(a) = 0$, then a is a global minimum of g , so $g'(a) = 0$, and

$$\frac{g(x)^{3/2} - 0}{x - a} = \frac{g(x)}{x - a} \cdot g(x)^{1/2} \longrightarrow 0 \cdot 0 = 0.$$

(c) Differentiability at a implies continuity at a .

(d) False. $f(x) = x$ and $a = 0$: $f^2 = x^2$ is differentiable at 0, but $|f| = |x|$ is not. The outer function $u^{1/2}$ is not differentiable at 0.

(e) False. Let $f = 1$ on the rationals and $f = -1$ on the irrationals. Then $f^2 \equiv 1$ is differentiable everywhere, but f is nowhere continuous.

(f) False. Let $f(x) = x$ for x rational and $f(x) = 0$ for x irrational, and take $a = 0$. Then $(f(x))^2 = x^2$ for x rational and 0 for x irrational, so

$$\left| \frac{(f(x))^2 - 0}{x - 0} \right| \leq |x| \longrightarrow 0,$$

giving $(f^2)'(0) = 0$. At any $b \neq 0$, however, f^2 takes the value 0 and values arbitrarily close to b^2 at points arbitrarily close to b , so f^2 is not continuous at b . Hence f^2 is differentiable at exactly one point.

3. Let f and g be functions defined on $\mathbb{R}$. Suppose that f is continuous at 0 and g is discontinuous at 0. Which of the following statements must be true? Select all that apply.

(a) If g is bounded on some open interval containing 0 and $f(0) = 0$, then fg is continuous at 0.

(b) If $f(0) \neq 0$, then fg is discontinuous at 0.

(c) f is bounded on some open interval containing 0.

(d) $f \circ g$ is discontinuous at 0.

(e) $\lim_{x \rightarrow 0} g(x)$ exists and is different from $g(0)$.

(f) $f + g$ is continuous at 0.

Solution:

- (a) If $|g| \leq M$ near 0, then $-M|f(x)| \leq f(x)g(x) \leq M|f(x)|$. Both bounds tend to $M|f(0)| = 0 = f(0)g(0)$, so the Squeeze Theorem gives continuity.
 - (b) Since f is continuous at 0 with $f(0) \neq 0$, f is nonzero on an open interval containing 0. If fg were continuous at 0, then $g = (fg)/f$ would be continuous at 0 by the quotient rule, a contradiction.
 - (c) Apply the definition of continuity at 0 with $\varepsilon = 1$: there is $\delta > 0$ with $|f(x) - f(0)| < 1$ for $|x| < \delta$, so $|f(x)| < |f(0)| + 1$ on $(-\delta, \delta)$.
 - (d) False. Take $f \equiv 7$ and $g(x) = \lfloor x \rfloor$. Then $f \circ g \equiv 7$.
 - (e) False. Discontinuity can also arise because the limit fails to exist. Take $g(x) = \lfloor x \rfloor$, whose one-sided limits at 0 are 0 and -1 .
 - (f) False. If $f + g$ were continuous at 0, then $g = (f + g) - f$ would be continuous at 0.
4. Let f be continuous on $\mathbb{R}$ with $f(x) \neq 0$ for every $x \in \mathbb{R}$. Which of the following statements must be true? Select all that apply.
- (a) Either $f(x) > 0$ for every $x \in \mathbb{R}$, or $f(x) < 0$ for every $x \in \mathbb{R}$.
 - (b) For every $p < q$, there exists $m > 0$ such that $|f(x)| \geq m$ for all $x \in [p, q]$.
 - (c) The function $\frac{1}{f}$ is continuous on $\mathbb{R}$.
 - (d) f is bounded on $\mathbb{R}$.
 - (e) f attains an absolute minimum value on $\mathbb{R}$.
 - (f) f is either increasing on $\mathbb{R}$ or decreasing on $\mathbb{R}$.

Solution:

- (a) If f took both a positive and a negative value, the Intermediate Value Theorem would give a zero on the interval between them.
- (b) By (a) the sign of f is constant. By the Extreme Value Theorem f attains extreme values on $[p, q]$. If $f > 0$, let m be the minimum value, which is positive; if $f < 0$, let m be the absolute value of the maximum value.
- (c) Quotient of continuous functions, with denominator nowhere zero.
- (d) False. $f(x) = x^2 + 1$.
- (e) False. $f(x) = \frac{1}{x^2 + 1}$ is continuous and never zero, with infimum 0 never attained. The Extreme Value Theorem needs a finite closed interval, and boundedness is not a substitute.
- (f) False. $f(x) = \sin(x) + 2$.

5. Let f and g be functions defined on an open interval containing 0, except possibly at 0, and suppose that $\lim_{x \rightarrow 0} f(x) = \infty$. Which of the following statements must be true? Select all that apply.

- (a) $\lim_{x \rightarrow 0} (f(x))^2 = \infty$.
- (b) If $f(x) \leq g(x)$ for all $x \neq 0$ in some open interval containing 0, then $\lim_{x \rightarrow 0} g(x) = \infty$.
- (c) f is not bounded above on any open interval containing 0.
- (d) If $\lim_{x \rightarrow 0} g(x) = -\infty$, then $\lim_{x \rightarrow 0} (f(x) + g(x))$ does not exist.
- (e) $f(x) > 0$ for every x in the domain of f .
- (f) The function h defined by $h(x) = \frac{1}{f(x)}$ for $x \neq 0$ and $h(0) = 0$ is differentiable at 0.

Solution:

- (a) Let $M > 0$. Apply the definition with $M' = \max\{1, \sqrt{M}\}$ to obtain $\delta > 0$ with $f(x) > M'$ on $0 < |x| < \delta$; then $(f(x))^2 > (M')^2 \geq M$.
 - (b) Let $M > 0$. On the intersection of the two neighbourhoods, $g(x) \geq f(x) > M$.
 - (c) Given any open interval I containing 0 and any $M > 0$, the definition provides $\delta > 0$ with $f(x) > M$ whenever $0 < |x| < \delta$. Shrinking δ so that $(-\delta, \delta) \subseteq I$, such x lie in I .
 - (d) False. Take $f(x) = \frac{1}{x^2} + 5$ and $g(x) = -\frac{1}{x^2}$ on $(-1, 1) \setminus \{0\}$. Then $f + g \equiv 5$.
 - (e) False. The conclusion concerns only x near 0. Take $f(x) = \frac{1}{x^2} - 1$ on $(-3, 3) \setminus \{0\}$; then $f(2) = -\frac{3}{4} < 0$.
 - (f) False. Take $f(x) = \frac{1}{|x|}$, so $h(x) = |x|$, which is not differentiable at 0.
6. Let f be continuous on $[1, 5]$ and differentiable on $(1, 5)$, with $f(1) = f(5) = 3$. Which of the following statements must be true? Select all that apply.
- (a) If $f(3) = 3$, then there exist $c_1, c_2 \in (1, 5)$, $c_1 \neq c_2$ with $f'(c_1) = f'(c_2) = 0$.
 - (b) If $f(4) = 9$, then there exists $c \in (1, 5)$ with $f'(c) = 0$ and $f(c) \geq 9$.
 - (c) There exists $c \in (1, 5)$ with $f'(c) = 0$.
 - (d) If $f(4) = 9$, then there exist $c_1, c_2 \in (1, 5)$, $c_1 \neq c_2$ with $f'(c_1) = f'(c_2) = 0$.
 - (e) There exists $c \in (1, 5)$ with $f(c) = 3$.

Question 6 continues...

(f) $f'(3) = 0$.

Solution:

- (a) Rolle's Theorem on $[1, 3]$ gives $c_1 \in (1, 3)$, and on $[3, 5]$ gives $c_2 \in (3, 5)$. The intervals are disjoint, so $c_1 \neq c_2$.
- (b) Rolle's Theorem is not enough here. By the Extreme Value Theorem f attains an absolute maximum value M on $[1, 5]$, and $M \geq f(4) = 9 > 3$, so the maximum is not attained at either endpoint. Hence it is attained at an interior point c , where f is differentiable, and Fermat's Theorem gives $f'(c) = 0$ with $f(c) = M \geq 9$.
- (c) Rolle's Theorem on $[1, 5]$, using $f(1) = f(5)$.
- (d) False, and (f) is false for the same function. Let

$$f(x) = \begin{cases} 9 - \frac{2}{3}(x-4)^2, & 1 \leq x \leq 4, \\ 9 - 6(x-4)^2, & 4 < x \leq 5. \end{cases}$$

Then $f(1) = f(5) = 3$, $f(4) = 9$, and the one-sided derivatives at 4 are both 0, so f is differentiable on $(1, 5)$. Here f' vanishes only at $x = 4$.

- (e) False. $f(x) = 3 + (x-1)(5-x)$ satisfies the hypotheses and is strictly greater than 3 on $(1, 5)$.
- (f) False, by the function in (d): $f'(3) = \frac{4}{3}$.

7. Let f and g be functions defined on $\mathbb{R}$, and let $a \in \mathbb{R}$. Which of the following statements must be true? Select all that apply.

- (a) If g is constant on some open interval containing a , then $f \circ g$ is differentiable at a .
- (b) If f is twice differentiable at a , then f' is continuous at a .
- (c) If f and g are both differentiable at a , then fg is differentiable at a .
- (d) If $f \circ g$ is differentiable at a , then g is differentiable at a .
- (e) If neither f nor g is differentiable at a , then fg is not differentiable at a .
- (f) If f and g are both differentiable at a , then $f \circ g$ is differentiable at a .

Solution:

- (a) $f \circ g$ is constant on that interval, and a constant function is differentiable.
- (b) $f''(a)$ exists means f' is differentiable at a , and differentiability at a point implies continuity at that point.

- (c) The Product Rule.
- (d) False. $f \equiv 0$ and $g(x) = |x|$ at $a = 0$: $f \circ g \equiv 0$.
- (e) False. $f(x) = g(x) = |x|$ at $a = 0$: $fg(x) = x^2$.
- (f) False. The Chain Rule requires f to be differentiable at $g(a)$, not at a . Take $a = 0$, $g(x) = x + 1$ and $f(x) = |x - 1|$. Both are differentiable at 0, but $f(g(x)) = |x|$. Contrast with (c): the product of two functions differentiable at a is always differentiable at a , the composite is not.
8. Let f be continuous on $[0, 10]$ and differentiable on $(0, 10)$, with $f(0) = 0$ and $|f'(x)| \leq 2$ for all $x \in (0, 10)$. Which of the following statements must be true? Select all that apply.
- (a) $|f(x)| \leq 2x$ for every $x \in [0, 10]$.
- (b) If $f(10) = 20$, then $f'(x) = 2$ for every $x \in (0, 10)$.
- (c) There exists $c \in (0, 10)$ with $f'(c) = \frac{f(10)}{10}$.
- (d) If $f(10) = 10$, then $f'(x) = 1$ for every $x \in (0, 10)$.
- (e) f has at most one zero in $(0, 10]$.
- (f) $f(x) \geq 0$ for every $x \in [0, 10]$.

Solution:

- (a) For $x \in (0, 10]$, the Mean Value Theorem on $[0, x]$ gives c with $f(x) - f(0) = f'(c)x$, so $|f(x)| \leq 2x$. The case $x = 0$ is immediate.
- (b) Fix $x \in (0, 10)$. By (a), $f(x) \leq 2x$. Applying the Mean Value Theorem on $[x, 10]$ gives $20 - f(x) = f'(d)(10 - x) \leq 2(10 - x)$, hence $f(x) \geq 2x$. So $f(x) = 2x$ on $[0, 10]$ and $f' \equiv 2$.
- (c) The Mean Value Theorem on $[0, 10]$, with $f(0) = 0$.
- (d) False. The bound is not saturated here, so f is not rigid. Take $f(x) = 2x$ on $[0, 5]$ and $f(x) = 10$ on $[5, 10]$.
- (e) False. $f(x) = \sin x$ satisfies every hypothesis and vanishes at π , 2π and 3π , all in $(0, 10]$.
- (f) False. $f(x) = -x$.
9. Let f be differentiable on the open interval $(0, 1)$. Which of the following statements must be true? Select all that apply.
- (a) If $f'(x) \geq 0$ for every $x \in (0, 1)$ and $f'(x) = 0$ for at most finitely many $x \in (0, 1)$, then f is strictly increasing on $(0, 1)$.

Question 9 continues...

- (b) If f is strictly increasing on $(0, 1)$, then $f'(x) \geq 0$ for every $x \in (0, 1)$.
- (c) If $f'(x) > 0$ for every $x \in (0, 1)$, then f is strictly increasing on $(0, 1)$.
- (d) If $f'(x) \geq 0$ for every $x \in (0, 1)$, then f is strictly increasing on $(0, 1)$.
- (e) If f is strictly increasing on $(0, 1)$, then $f'(x) > 0$ for every $x \in (0, 1)$.
- (f) If $f'\left(\frac{1}{2}\right) > 0$, then $f(x) > f\left(\frac{1}{2}\right)$ for every $x \in \left(\frac{1}{2}, 1\right)$.

Solution:

- (a) Let $p < q$ in $(0, 1)$. Choose a subinterval $[r, s] \subset (p, q)$ containing none of the finitely many zeros; the Increasing Test gives $f(r) < f(s)$. On $[p, r]$ and $[s, q]$, the Mean Value Theorem with $f' \geq 0$ gives $f(p) \leq f(r)$ and $f(s) \leq f(q)$. Chaining, $f(p) < f(q)$.
 - (b) For $x \neq a$ the difference quotient $\frac{f(x) - f(a)}{x - a}$ is positive, so its limit $f'(a)$ is ≥ 0 .
 - (c) The Increasing Test.
 - (d) False. A constant function has $f' \equiv 0$ and is not strictly increasing. Options (a), (c) and (d) mark the exact boundary: strict positivity suffices, non-negativity does not, and non-negativity with finitely many zeros does.
 - (e) False. $f(x) = \left(x - \frac{1}{2}\right)^3$ is strictly increasing with $f'\left(\frac{1}{2}\right) = 0$.
 - (f) False. The sign of f' at one point controls nothing on an interval. Take $f(x) = -(x - 0.6)^2$, so $f'(0.5) = 0.2 > 0$, yet $f(0.9) = -0.09 < -0.01 = f(0.5)$.
10. Let f be defined on $(-1, 1)$ and differentiable at every point of $(-1, 1)$ except possibly at 0. Suppose that $f'(x) > 0$ for all $x \in (-1, 0)$ and $f'(x) < 0$ for all $x \in (0, 1)$. Which of the following statements must be true? Select all that apply.
- (a) f is strictly increasing on $(-1, 0)$ and strictly decreasing on $(0, 1)$.
 - (b) If f is continuous at 0, then $f(0) \geq f(x)$ for every $x \in (-1, 1)$.
 - (c) f is continuous at every $x \in (-1, 1)$ with $x \neq 0$.
 - (d) f is continuous at 0.
 - (e) $f'(0) = 0$.
 - (f) f has an absolute maximum value on $(-1, 1)$.

Solution:

- (a) On each of $(-1, 0)$ and $(0, 1)$, f is differentiable, hence continuous, and the Increasing and Decreasing Tests apply. Nothing about the point 0 is required.

- (b) For $x \in (-1, 0)$ and $x < t < 0$, (a) gives $f(x) < f(t)$; letting $t \rightarrow 0^-$ and using continuity, $f(x) \leq f(0)$. The same argument on the right gives $f(x) \leq f(0)$ for $x \in (0, 1)$.
- (c) Differentiability at a point implies continuity at that point, and f is differentiable at every such x .
- (d) False, and (f) is false for the same function. Let $f(x) = x$ on $(-1, 0)$, $f(0) = -5$, and $f(x) = -x$ on $(0, 1)$. The hypotheses hold, f is not continuous at 0, and $\sup f = 0$ is never attained.
- (e) False. $f(x) = -|x|$ satisfies the hypotheses and $f'(0)$ does not exist.
- (f) False, by the function in (d). This is why the First Derivative Test assumes continuity at the critical point.
11. Let f be differentiable on $\mathbb{R}$ with $f'(2) = 0$, and suppose that $f''(2)$ exists. Which of the following statements must be true? Select all that apply.
- (a) If $f''(2) < 0$, then there exists $\delta > 0$ such that $f(x) < f(2)$ for all x with $0 < |x - 2| < \delta$.
- (b) If $f''(2) > 0$, then there exists $\delta > 0$ such that $f'(x) > 0$ for all $x \in (2, 2 + \delta)$.
- (c) If $f''(2) > 0$, then $f(2)$ is a local minimum value of f .
- (d) If $f''(2) = 0$, then $f(2)$ is not a local extreme value of f .
- (e) If $f(2)$ is a local maximum value of f , then $f''(2) < 0$.
- (f) If $f''(2) < 0$, then $f''(x) < 0$ for all x in some open interval containing 2.

Solution: Since $f'(2) = 0$, we have $\frac{f'(x)}{x - 2} = \frac{f'(x) - f'(2)}{x - 2} \rightarrow f''(2)$.

- (a) If $f''(2) < 0$, the quotient is negative on a punctured interval, so $f' > 0$ on $(2 - \delta, 2)$ and $f' < 0$ on $(2, 2 + \delta)$. Hence f is strictly increasing then strictly decreasing, and the inequality is strict.
- (b) If $f''(2) > 0$, the same quotient is positive on a punctured interval, so $f'(x) > 0$ for $x \in (2, 2 + \delta)$.
- (c) The Second Derivative Test, since $f'(2) = 0$ and $f''(2) > 0$.
- (d) False. $f(x) = (x - 2)^4$ has $f'(2) = f''(2) = 0$ and a local minimum at 2.
- (e) False. $f(x) = -(x - 2)^4$ has a local maximum at 2 with $f''(2) = 0$.
- (f) False. $f''(2)$ is a statement about f' at the single point 2; f'' need not exist elsewhere. Writing $u = x - 2$, take $f'(x) = -u + u^2 \sin(1/u^2)$ for $u \neq 0$ and $f'(2) = 0$. Then $f''(2) = -1 < 0$, while for $u \neq 0$, $f''(x) = -1 + 2u \sin(1/u^2) - \frac{2}{u} \cos(1/u^2)$, which changes sign in every interval around 2.

12. Let f and g be functions defined on an open interval containing 2, except possibly at 2. Which of the following statements must be true? Select all that apply.

- (a) If $\lim_{x \rightarrow 2} f(x) = 0$ and g is bounded on some open interval containing 2, except possibly at 2, then $\lim_{x \rightarrow 2} f(x)g(x) = 0$.
- (b) If $\lim_{x \rightarrow 2} f(x)g(x) = 6$ and $\lim_{x \rightarrow 2} f(x) = 2$, then $\lim_{x \rightarrow 2} g(x) = 3$.
- (c) If $\lim_{x \rightarrow 2} f(x) = 3$, then $\lim_{x \rightarrow 2} (f(x))^2 = 9$.
- (d) If $\lim_{x \rightarrow 2} f(x)g(x)$ exists and $\lim_{x \rightarrow 2} g(x)$ exists, then $\lim_{x \rightarrow 2} f(x)$ exists.
- (e) If $\lim_{x \rightarrow 2} (f(x))^2 = 9$, then $\lim_{x \rightarrow 2} f(x) = 3$ or $\lim_{x \rightarrow 2} f(x) = -3$.
- (f) If $\lim_{x \rightarrow 2} f(x)g(x) = 0$, then $\lim_{x \rightarrow 2} f(x) = 0$ or $\lim_{x \rightarrow 2} g(x) = 0$.

Solution:

- (a) If $|g| \leq M$ near 2, then $-M|f| \leq fg \leq M|f|$ and the Squeeze Theorem applies.
- (b) Since $\lim_{x \rightarrow 2} f(x) = 2 \neq 0$, f is nonzero near 2 and the quotient law gives $\lim_{x \rightarrow 2} g(x) = \lim_{x \rightarrow 2} \frac{f(x)g(x)}{f(x)} = \frac{6}{2} = 3$.
- (c) The Product Law.
- (d) False. Take $g \equiv 0$ and $f(x) = \frac{1}{x-2}$. Then $fg \equiv 0$, but $\lim_{x \rightarrow 2} f(x)$ does not exist.
- (e) False, and note that it is the converse of (c). Take $f(x) = 3$ for $x > 2$ and $f(x) = -3$ for $x < 2$. Then $f^2 \equiv 9$, but $\lim_{x \rightarrow 2} f(x)$ does not exist.
- (f) False. Take $f(x) = x - 2$ and $g(x) = 1$ for $x > 2$, and $f(x) = 1$ and $g(x) = x - 2$ for $x < 2$. Then $f(x)g(x) = x - 2 \rightarrow 0$, while neither $\lim_{x \rightarrow 2} f(x)$ nor $\lim_{x \rightarrow 2} g(x)$ exists.