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Department of Mathematics

MA2002 Calculus **Tutorial 3**

1. Using
$$\lim_{\theta\to 0}\frac{\sin\theta}{\theta}=1,$$
find the following limits.

(a) $\lim\limits_{x\to -1}\dfrac{\sin(x+1)}{x^2-x-2}$.

Solution: Factor the denominator: $x^2-x-2=(x-2)(x+1)$, so
$$\frac{\sin(x+1)}{x^2-x-2}=\frac{\sin(x+1)}{x+1}\cdot\frac{1}{x-2}.$$
As $x\to -1$, let $u=x+1\to 0$. Then $\dfrac{\sin(x+1)}{x+1}=\dfrac{\sin u}{u}\to 1$, and $\dfrac{1}{x-2}\to\dfrac{1}{-1-2}=-\dfrac{1}{3}$. Hence
$$\lim_{x\to -1}\frac{\sin(x+1)}{x^2-x-2}=1\cdot\left(-\frac{1}{3}\right)=-\frac{1}{3}.$$

(b) $\lim\limits_{x\to 0}\dfrac{\sin ax}{\tan bx}$ $\quad(a\neq 0,\ b\neq 0)$.

Solution: Write $\tan bx=\dfrac{\sin bx}{\cos bx}$, so
$$\frac{\sin ax}{\tan bx}=\frac{\sin ax\cos bx}{\sin bx}=\left(\frac{\sin ax}{ax}\right)\left(\frac{bx}{\sin bx}\right)\left(\frac{a}{b}\right)\cos bx.$$
As $x\to 0$, $ax\to 0$ and $bx\to 0$, so $\dfrac{\sin ax}{ax}\to 1$ and $\dfrac{bx}{\sin bx}\to 1$; also $\cos bx\to 1$. Therefore
$$\lim_{x\to 0}\frac{\sin ax}{\tan bx}=1\cdot 1\cdot\frac{a}{b}\cdot 1=\frac{a}{b}.$$

2. Find $\dfrac{d^2y}{dx^2}$ for the function $y=\sin(\cos x)$.

Solution: By the chain rule,
$$\frac{dy}{dx}=\cos(\cos x)\cdot(-\sin x)=-\sin x\cos(\cos x).$$

Differentiate again using the product rule:

$$\begin{aligned}\frac{d^2y}{dx^2} &= -\frac{d}{dx}(\sin x)\cdot\cos(\cos x) - \sin x\cdot\frac{d}{dx}(\cos(\cos x))\\ &= (-\cos x)\cos(\cos x) + (-\sin x)\big(\sin x\sin(\cos x)\big)\\ &= -\cos x\cos(\cos x) - \sin^2 x\sin(\cos x).\end{aligned}$$

3. Find $\dfrac{dy}{dx}$ by implicit differentiation:

   (a) $\sin(xy) = x + y$.

   Solution: Differentiating both sides with respect to x:

$$\cos(xy)\cdot\left(y + x\frac{dy}{dx}\right) = 1 + \frac{dy}{dx}.$$

   Expand and collect terms in $\dfrac{dy}{dx}$:

$$y\cos(xy) + x\frac{dy}{dx}\cos(xy) - \frac{dy}{dx} = 1 \implies \frac{dy}{dx}\big[x\cos(xy) - 1\big] = 1 - y\cos(xy).$$

   Hence

$$\frac{dy}{dx} = \frac{1 - y\cos(xy)}{x\cos(xy) - 1}.$$

   (b) $x^3 + y^3 = 3xy$.

   Solution: Differentiating both sides with respect to x:

$$3x^2 + 3y^2\frac{dy}{dx} = 3y + 3x\frac{dy}{dx}.$$

   Collecting terms in $\dfrac{dy}{dx}$:

$$3y^2\frac{dy}{dx} - 3x\frac{dy}{dx} = 3y - 3x^2 \implies \frac{dy}{dx}(y^2 - x) = y - x^2,$$

   so

$$\frac{dy}{dx} = \frac{y - x^2}{y^2 - x} \qquad (y^2 \neq x).$$

4. Let (x_0, y_0) be a point on the hyperbola

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1.$$

Show that the tangent line to the hyperbola passing through (x_0, y_0) is

$$\frac{x_0 x}{a^2} - \frac{y_0 y}{b^2} = 1.$$

Solution: Differentiating $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ implicitly:

$$\frac{2x}{a^2} - \frac{2y\frac{dy}{dx}}{b^2} = 0 \implies \frac{dy}{dx} = \frac{b^2 x}{a^2 y} \quad (y \neq 0).$$

Case $y_0 \neq 0$. The tangent slope at (x_0, y_0) is $\frac{b^2 x_0}{a^2 y_0}$, so the tangent line is

$$y - y_0 = \frac{b^2 x_0}{a^2 y_0}(x - x_0).$$

Multiply both sides by $a^2 y_0$:

$$a^2 y_0 y - a^2 y_0^2 = b^2 x_0 x - b^2 x_0^2 \implies b^2 x_0 x - a^2 y_0 y = b^2 x_0^2 - a^2 y_0^2.$$

Since (x_0, y_0) lies on the hyperbola, $\frac{x_0^2}{a^2} - \frac{y_0^2}{b^2} = 1$, so multiplying by $a^2 b^2$ gives $b^2 x_0^2 - a^2 y_0^2 = a^2 b^2$. Thus

$$b^2 x_0 x - a^2 y_0 y = a^2 b^2 \implies \frac{x_0 x}{a^2} - \frac{y_0 y}{b^2} = 1.$$

Case $y_0 = 0$. Then $x_0 = \pm a$, and by symmetry the tangent line at each vertex is vertical: $x = x_0$. Substituting $y_0 = 0$ into the claimed formula gives $\frac{x_0 x}{a^2} = 1 \implies x = \frac{a^2}{x_0} = x_0$ (since $x_0 = \pm a$), which agrees.

Hence the tangent line formula holds for every point on the hyperbola.

5. Use the closed interval method to find the absolute maximum and minimum values of each function f on the given interval and the values of x where they occur.

   (a) $f(x) = x^3 - 3x^2 - 9x + 5, \qquad [-2, 5].$

   Solution: $f'(x) = 3x^2 - 6x - 9 = 3(x-3)(x+1)$, so the critical points are $x = -1$ and $x = 3$, both in $[-2, 5]$. Evaluating:

   $$f(-2) = 3, \qquad f(-1) = 10, \qquad f(3) = -22, \qquad f(5) = 10.$$

   The absolute maximum value is 10, attained at both $x = -1$ and $x = 5$. The absolute minimum value is -22, attained at $x = 3$.

(b) $f(x) = \sqrt[3]{x}\,(x-4), \qquad [-1,5].$

Solution: Write $f(x) = x^{4/3} - 4x^{1/3}$. Then

$$f'(x) = \frac{4}{3}x^{1/3} - \frac{4}{3}x^{-2/3} = \frac{4}{3}x^{-2/3}(x-1).$$

f' is undefined at $x = 0$, so $x = 0$ is a critical point; $f'(x) = 0$ at $x = 1$, so $x = 1$ is also a critical point. Both lie in $[-1, 5]$. Evaluating:

$$f(-1) = 5, \qquad f(0) = 0, \qquad f(1) = -3, \qquad f(5) = \sqrt[3]{5}.$$

The absolute maximum value is 5, attained at $x = -1$. The absolute minimum value is -3, attained at $x = 1$.

6. Let $r > 1$ be a rational number. Prove that for any $x \in [0, 1]$,

$$0 \le x^r(1-x)^r \le \frac{1}{4^r}.$$

Solution: Let $h(x) = x^r(1-x)^r$ for $x \in [0,1]$; h is continuous on $[0,1]$. For $x \in (0,1)$,

$$h'(x) = rx^{r-1}(1-x)^r - rx^r(1-x)^{r-1} = rx^{r-1}(1-x)^{r-1}\big[(1-x)-x\big] = rx^{r-1}(1-x)^{r-1}(1-2x).$$

Since $r - 1 > 0$, both $x^{r-1} > 0$ and $(1-x)^{r-1} > 0$ on $(0,1)$, so $h'(x) = 0$ if and only if $x = \frac{1}{2}$; this is the only critical point in $(0,1)$.
Evaluating h at the endpoints and the critical point:

$$h(0) = 0, \qquad h(1) = 0, \qquad h\left(\frac{1}{2}\right) = \left(\frac{1}{2}\right)^r\left(\frac{1}{2}\right)^r = \left(\frac{1}{2}\right)^{2r} = \frac{1}{4^r}.$$

By the closed interval method, the absolute minimum of h on $[0,1]$ is 0 and the absolute maximum is $\frac{1}{4^r}$. Therefore

$$0 \le x^r(1-x)^r \le \frac{1}{4^r} \qquad \text{for all } x \in [0,1].$$

7. Let $f(x) = 2x^3 + bx^2 + cx + d$ be a cubic function. Show that if $b^2 < 6c$, then f has no local extreme values.

Solution: We have $f'(x) = 6x^2 + 2bx + c$, a quadratic with leading coefficient $6 > 0$ and discriminant

$$\Delta = (2b)^2 - 4(6)(c) = 4b^2 - 24c = 4(b^2 - 6c).$$

If $b^2 < 6c$, then $\Delta < 0$, so f' has no real roots; that is, $f'(x) \neq 0$ for all $x \in \mathbb{R}$.

Since f' is a continuous function that is never zero, by the Intermediate Value Theorem f' cannot change sign — if it were negative somewhere and positive elsewhere, IVT would force a root in between. As $f'(x) \to +\infty$ as $x \to \infty$, we conclude $f'(x) > 0$ for all x. Hence f is strictly increasing on $\mathbb{R}$.

By Fermat's Theorem, any local extreme value of f must occur at a critical point (where $f' = 0$). Since f' is never zero, f has no critical points, and therefore no local extreme values.

8. Suppose that f is differentiable on $\mathbb{R}$ and that $f'(x) \neq 2$ for all $x \in \mathbb{R}$. Prove that the equation $f(x) = 2x + 5$ has *at most one* solution in $\mathbb{R}$.

   Solution: Suppose, for contradiction, that the equation has two distinct solutions $a < b$. Then $f(a) = 2a + 5$ and $f(b) = 2b + 5$.

   Since f is differentiable on $\mathbb{R}$, it is differentiable on (a, b) and continuous on $[a, b]$. By the Mean Value Theorem, there exists $c \in (a, b)$ such that

$$f'(c) = \frac{f(b) - f(a)}{b - a} = \frac{(2b + 5) - (2a + 5)}{b - a} = \frac{2(b - a)}{b - a} = 2.$$

   This contradicts the hypothesis that $f'(x) \neq 2$ for all x. Hence no two distinct solutions can exist, and $f(x) = 2x + 5$ has at most one solution in $\mathbb{R}$.

9. Suppose that f is a function such that

$$f'(x) = \frac{2}{x}$$

   for all $x > 0$. Prove that if $f(1) = 0$, then

$$f(x^2 y) = 2f(x) + f(y)$$

   for all $x > 0$ and $y > 0$.

   Solution: Fix $y > 0$ and define, for $x > 0$,

$$g(x) = f(x^2 y) - 2f(x) - f(y).$$

   Differentiating with respect to x (treating y as constant), and using the chain rule on the first term:

$$g'(x) = f'(x^2 y) \cdot \frac{d}{dx}(x^2 y) - 2f'(x) = 2xy\, f'(x^2 y) - 2f'(x).$$

   Since $x > 0$ and $y > 0$ imply $x^2 y > 0$, we may substitute $f'(t) = \dfrac{2}{t}$ for both $t = x^2 y$ and $t = x$:

$$g'(x) = 2xy \cdot \frac{2}{x^2 y} - 2 \cdot \frac{2}{x} = \frac{4}{x} - \frac{4}{x} = 0.$$

So $g'(x) = 0$ for all $x > 0$, meaning g is constant on $(0, \infty)$.
Evaluating at $x = 1$:

$$g(1) = f(y) - 2f(1) - f(y) = -2f(1) = -2(0) = 0,$$

using $f(1) = 0$. Since g is constant and $g(1) = 0$, we have $g(x) = 0$ for all $x > 0$; that is,

$$f(x^2y) - 2f(x) - f(y) = 0 \implies f(x^2y) = 2f(x) + f(y).$$

As $y > 0$ was arbitrary, this holds for all $x > 0$ and $y > 0$.