

NATIONAL UNIVERSITY OF SINGAPORE
Department of Mathematics

MA2002 Calculus

Tutorial 4

1. Evaluate the following limits, using l'Hôpital's Rule where appropriate.

(a) $\lim_{x \rightarrow 0} \frac{e^x - 1 - x}{x^2}.$

Solution: As $x \rightarrow 0$, numerator and denominator both $\rightarrow 0$, so the form is $\frac{0}{0}$. By l'Hôpital,

$$\lim_{x \rightarrow 0} \frac{e^x - 1 - x}{x^2} = \lim_{x \rightarrow 0} \frac{e^x - 1}{2x}.$$

This is again $\frac{0}{0}$; applying l'Hôpital once more,

$$= \lim_{x \rightarrow 0} \frac{e^x}{2} = \frac{1}{2}.$$

(b) $\lim_{x \rightarrow 0} \frac{\tan x - x}{x^3}.$

Solution: The form is $\frac{0}{0}$. By l'Hôpital, using $\frac{d}{dx} \tan x = \sec^2 x$,

$$\lim_{x \rightarrow 0} \frac{\tan x - x}{x^3} = \lim_{x \rightarrow 0} \frac{\sec^2 x - 1}{3x^2} = \lim_{x \rightarrow 0} \frac{\tan^2 x}{3x^2},$$

using $\sec^2 x - 1 = \tan^2 x$. Since $\frac{\tan x}{x} \rightarrow 1$,

$$= \frac{1}{3} \lim_{x \rightarrow 0} \left(\frac{\tan x}{x} \right)^2 = \frac{1}{3}.$$

(c) $\lim_{x \rightarrow \infty} \frac{\ln x}{\sqrt{x}}.$

Solution: As $x \rightarrow \infty$, the form is $\frac{\infty}{\infty}$. By l'Hôpital,

$$\lim_{x \rightarrow \infty} \frac{\ln x}{\sqrt{x}} = \lim_{x \rightarrow \infty} \frac{1/x}{\frac{1}{2}x^{-1/2}} = \lim_{x \rightarrow \infty} \frac{2}{\sqrt{x}} = 0.$$

(d) $\lim_{x \rightarrow (\pi/2)^-} (\sec x - \tan x).$

Solution: This is an $\infty - \infty$ form. Combine over a common denominator:

$$\sec x - \tan x = \frac{1}{\cos x} - \frac{\sin x}{\cos x} = \frac{1 - \sin x}{\cos x}.$$

As $x \rightarrow (\pi/2)^-$, this is $\frac{0}{0}$; by l'Hôpital,

$$\lim_{x \rightarrow (\pi/2)^-} \frac{1 - \sin x}{\cos x} = \lim_{x \rightarrow (\pi/2)^-} \frac{-\cos x}{-\sin x} = \frac{0}{-1} = 0.$$

2. Let $f(x) = \frac{x^2}{x^2 - 1}$.

- (a) State the domain of f and find all intercepts and asymptotes (vertical and horizontal).

Solution: The domain is all x with $x^2 - 1 \neq 0$, i.e. $x \neq \pm 1$. Setting $x = 0$ gives $f(0) = 0$, so the only intercept is the origin.

As $x \rightarrow \pm 1$, the denominator $\rightarrow 0$ while the numerator $\rightarrow 1$, so $x = 1$ and $x = -1$ are vertical asymptotes. Since

$$\lim_{x \rightarrow \pm\infty} \frac{x^2}{x^2 - 1} = \lim_{x \rightarrow \pm\infty} \frac{1}{1 - 1/x^2} = 1,$$

the line $y = 1$ is a horizontal asymptote.

- (b) Find the intervals on which f is increasing and decreasing, and all local extrema.

Solution: Differentiating,

$$f'(x) = \frac{2x(x^2 - 1) - x^2(2x)}{(x^2 - 1)^2} = \frac{-2x}{(x^2 - 1)^2}.$$

The denominator is positive on the domain, so the sign of f' is the sign of $-2x$. Hence $f'(x) > 0$ for $x < 0$ and $f'(x) < 0$ for $x > 0$ (excluding $x = \pm 1$):

- Increasing on $(-\infty, -1)$ and $(-1, 0)$;
- Decreasing on $(0, 1)$ and $(1, \infty)$.

At $x = 0$, f' changes from positive to negative, so f has a local maximum at $x = 0$ with value $f(0) = 0$.

- (c) Find the intervals of concavity, and determine whether f has any inflection points.

Solution: Differentiating again,

$$f''(x) = \frac{d}{dx} \left(\frac{-2x}{(x^2 - 1)^2} \right) = \frac{2(3x^2 + 1)}{(x^2 - 1)^3}.$$

The numerator $2(3x^2 + 1)$ is always positive, so the sign of f'' is the sign of $(x^2 - 1)^3$, i.e. of $x^2 - 1$:

- $x^2 - 1 > 0$ on $(-\infty, -1)$ and $(1, \infty)$: concave up;
- $x^2 - 1 < 0$ on $(-1, 1)$: concave down.

The concavity changes only across $x = \pm 1$, where f is undefined. Since an inflection point requires f to be continuous there, f has *no* inflection points.

3. (a) Among all rectangles with a fixed perimeter P , find the dimensions of the one with the largest area, using the closed interval method.

Solution: Let one side have length x ; since the perimeter is P , the adjacent side has length $\frac{P}{2} - x$. For the rectangle to be non-degenerate we need $0 \leq x \leq \frac{P}{2}$. The area is

$$A(x) = x \left(\frac{P}{2} - x \right) = \frac{P}{2}x - x^2, \quad x \in \left[0, \frac{P}{2} \right].$$

A is continuous on the closed bounded interval $\left[0, \frac{P}{2} \right]$, so by the closed interval method its absolute maximum occurs at a critical point or an endpoint. Differentiating,

$$A'(x) = \frac{P}{2} - 2x = 0 \implies x = \frac{P}{4},$$

the only critical point. Evaluating,

$$A(0) = 0, \quad A\left(\frac{P}{2}\right) = 0, \quad A\left(\frac{P}{4}\right) = \frac{P^2}{16}.$$

The absolute maximum is $\frac{P^2}{16}$, attained at $x = \frac{P}{4}$. The two sides are then both $\frac{P}{4}$, so the optimal rectangle is a square of side $\frac{P}{4}$.

(Remark: that the optimum is a square is expected from symmetry — neither side is distinguished, so the extremal rectangle should treat them equally.)

- (b) A closed rectangular box with a square base is to have a volume of 2000 cm^3 . Find the dimensions that minimise the total surface area.

Solution: Let the base have side s cm and the box have height h cm. The volume constraint is $s^2h = 2000$, so $h = \frac{2000}{s^2}$.

The closed box has a top and bottom (combined area $2s^2$) and four sides (combined area $4sh$), so the total surface area is

$$S = 2s^2 + 4sh = 2s^2 + 4s \cdot \frac{2000}{s^2} = 2s^2 + \frac{8000}{s}, \quad s > 0.$$

Then

$$S'(s) = 4s - \frac{8000}{s^2} = 0 \implies 4s^3 = 8000 \implies s^3 = 2000 \implies s = \sqrt[3]{2000} = 10\sqrt[3]{2}.$$

Since $S''(s) = 4 + \frac{16000}{s^3} > 0$ for all $s > 0$, this critical point is a minimum. The height is

$$h = \frac{2000}{s^2} = \frac{2000}{(2000)^{2/3}} = (2000)^{1/3} = 10\sqrt[3]{2} = s.$$

The surface area is minimised when $h = s$; that is, the optimal closed box is a cube of side $10\sqrt[3]{2} \approx 12.6$ cm.

- (c) Repeat part (b) for an *open-top* box (no top face), with the same volume. Compare the optimal shape with that in part (b).

Solution: With no top, the surface consists of the bottom (area s^2) and the four sides (area $4sh$). With $h = \frac{2000}{s^2}$,

$$S = s^2 + 4sh = s^2 + \frac{8000}{s}, \quad s > 0.$$

Then

$$S'(s) = 2s - \frac{8000}{s^2} = 0 \implies 2s^3 = 8000 \implies s^3 = 4000 \implies s = \sqrt[3]{4000} = 10\sqrt[3]{4}.$$

Since $S''(s) = 2 + \frac{16000}{s^3} > 0$ for all $s > 0$, this is a minimum. The height is

$$h = \frac{2000}{s^2} = \frac{2000}{(4000)^{2/3}}.$$

Writing $4000 = 2000 \cdot 2$, one finds

$$h = \frac{s}{2},$$

so the optimal open box has height equal to *half* the base side: $s = 10\sqrt[3]{4} \approx 15.9$ cm and $h \approx 7.9$ cm.

Compared with part (b), removing the top changes the optimal shape from a cube ($h = s$) to a flatter box ($h = \frac{1}{2}s$): with no top to enclose, it is more efficient to widen the base and lower the height.

(Remark: as in part (a), symmetry anticipates this — for a closed box no face is distinguished, so the surface-minimising shape should treat all three dimensions equally, giving a cube.)

4. Let $a > 0$. Using Riemann sums and the identity $\sum_{i=1}^n i^2 = \frac{n(n+1)(2n+1)}{6}$, compute $\int_0^a x^2 dx$ directly from the definition of the definite integral.

Solution: Partition $[0, a]$ into n equal subintervals of width $\Delta x = \frac{a}{n}$, with right endpoints $x_i = \frac{ia}{n}$ for $i = 1, \dots, n$. Since x^2 is continuous, the definite integral is the limit of right Riemann sums:

$$\int_0^a x^2 dx = \lim_{n \rightarrow \infty} \sum_{i=1}^n x_i^2 \Delta x = \lim_{n \rightarrow \infty} \sum_{i=1}^n \frac{i^2 a^2}{n^2} \cdot \frac{a}{n} = \lim_{n \rightarrow \infty} \frac{a^3}{n^3} \sum_{i=1}^n i^2.$$

Using the identity,

$$= \lim_{n \rightarrow \infty} \frac{a^3}{n^3} \cdot \frac{n(n+1)(2n+1)}{6} = \frac{a^3}{6} \lim_{n \rightarrow \infty} \frac{(n+1)(2n+1)}{n^2} = \frac{a^3}{6} \cdot 2 = \frac{a^3}{3}.$$

5. Let f be continuous on an interval I , and let $a, b, c \in I$. The additivity property

$$\int_a^c f(x) dx + \int_c^b f(x) dx = \int_a^b f(x) dx$$

holds for *any* $a, b, c \in I$, not only when c lies between a and b . Assuming the property for the case $a < c < b$, prove it in the case $a < b < c$ (so that c lies *outside* the interval $[a, b]$).

Solution: Since $a < b < c$, the point b lies between a and c , so the additivity property in the already-established order gives

$$\int_a^b f(x) dx + \int_b^c f(x) dx = \int_a^c f(x) dx.$$

Rearranging,

$$\int_a^b f(x) dx = \int_a^c f(x) dx - \int_b^c f(x) dx.$$

By the order-reversal property $\int_b^c f = -\int_c^b f$, so

$$\int_a^b f(x) dx = \int_a^c f(x) dx + \int_c^b f(x) dx.$$

This is exactly the additivity identity for the order $a < b < c$, with c lying outside $[a, b]$.

6. Find $\frac{dy}{dx}$ for each of the following.

(a) $y = \int_0^{x^2} \cos(t^{1/3}) dt.$

Solution: By the Fundamental Theorem of Calculus with the chain rule (upper limit x^2),

$$\frac{dy}{dx} = \cos((x^2)^{1/3}) \cdot \frac{d}{dx}(x^2) = 2x \cos(x^{2/3}).$$

(b) $y = \int_{\pi}^{\sqrt{x}} \sin t dt.$

Solution: By the FTC with the chain rule (upper limit $\sqrt{x}$),

$$\frac{dy}{dx} = \sin(\sqrt{x}) \cdot \frac{d}{dx}\sqrt{x} = \frac{\sin(\sqrt{x})}{2\sqrt{x}}.$$

(c) $y = \int_{\cos x}^{5x} \cos(t^2) dt.$

Solution: Both limits vary with x , so we split the integral at a constant. Pick any c in the interval (by Question 5 the additivity property holds for any such c):

$$y = \int_{\cos x}^{5x} \cos(t^2) dt = \int_{\cos x}^c \cos(t^2) dt + \int_c^{5x} \cos(t^2) dt.$$

Reverse the first integral so its variable limit is on top:

$$y = - \int_c^{\cos x} \cos(t^2) dt + \int_c^{5x} \cos(t^2) dt.$$

Now each integral has a constant lower limit, so the Fundamental Theorem of Calculus with the chain rule applies to each:

$$\frac{dy}{dx} = -\cos((\cos x)^2) \cdot \frac{d}{dx}(\cos x) + \cos((5x)^2) \cdot \frac{d}{dx}(5x).$$

Since $\frac{d}{dx}(\cos x) = -\sin x$ and $\frac{d}{dx}(5x) = 5$,

$$\frac{dy}{dx} = -\cos(\cos^2 x)(-\sin x) + 5 \cos(25x^2) = \sin x \cos(\cos^2 x) + 5 \cos(25x^2).$$

7. Let f be a continuous function on $\mathbb{R}$, and define

$$F(x) = \int_a^x f(t) (x - t) dt.$$

Evaluate $F''(x)$.

Solution: Since x appears both in the limit and in the integrand, first separate it. As x is constant with respect to t ,

$$F(x) = \int_a^x (xf(t) - tf(t)) dt = x \int_a^x f(t) dt - \int_a^x tf(t) dt.$$

Differentiate using the product rule on the first term and the FTC on both:

$$F'(x) = \int_a^x f(t) dt + xf(x) - xf(x) = \int_a^x f(t) dt.$$

Differentiating once more by the FTC,

$$F''(x) = f(x).$$

8. Evaluate the following integrals.

(a) $\int \frac{\cos(\pi/x)}{x^2} dx.$

Solution: Let $u = \frac{\pi}{x}$, so $du = -\frac{\pi}{x^2} dx$, giving $\frac{1}{x^2} dx = -\frac{1}{\pi} du$. Then

$$\int \frac{\cos(\pi/x)}{x^2} dx = -\frac{1}{\pi} \int \cos u du = -\frac{1}{\pi} \sin u + C = -\frac{1}{\pi} \sin \frac{\pi}{x} + C.$$

(b) $\int \frac{1}{\sqrt{x}(1+\sqrt{x})^2} dx.$

Solution: Let $u = 1 + \sqrt{x}$, so $du = \frac{1}{2\sqrt{x}} dx$, giving $\frac{1}{\sqrt{x}} dx = 2 du$. Then

$$\int \frac{1}{\sqrt{x}(1+\sqrt{x})^2} dx = \int \frac{2}{u^2} du = -\frac{2}{u} + C = -\frac{2}{1+\sqrt{x}} + C.$$

(c) $\int_0^1 x^2(x+1)^2 dx.$

Solution: Expand: $x^2(x+1)^2 = x^2(x^2+2x+1) = x^4+2x^3+x^2$. Then

$$\int_0^1 (x^4+2x^3+x^2) dx = \left[\frac{x^5}{5} + \frac{x^4}{2} + \frac{x^3}{3} \right]_0^1 = \frac{1}{5} + \frac{1}{2} + \frac{1}{3} = \frac{31}{30}.$$

(d) $\int_0^4 |\sqrt{x} - 1| dx.$

Solution: Since $\sqrt{x} - 1 \leq 0$ for $0 \leq x \leq 1$ and $\sqrt{x} - 1 \geq 0$ for $1 \leq x \leq 4$, split at $x = 1$:

$$\int_0^4 |\sqrt{x} - 1| dx = \int_0^1 (1 - \sqrt{x}) dx + \int_1^4 (\sqrt{x} - 1) dx.$$

Computing each,

$$\int_0^1 (1 - x^{1/2}) dx = \left[x - \frac{2}{3} x^{3/2} \right]_0^1 = 1 - \frac{2}{3} = \frac{1}{3},$$

$$\int_1^4 (x^{1/2} - 1) dx = \left[\frac{2}{3} x^{3/2} - x \right]_1^4 = \left(\frac{16}{3} - 4 \right) - \left(\frac{2}{3} - 1 \right) = \frac{4}{3} + \frac{1}{3} = \frac{5}{3}.$$

The total is $\frac{1}{3} + \frac{5}{3} = 2$.

9. Find the area of the region between the graph of each function and the x -axis over the given interval.

(a) $y = x^3 - 3x^2 + 2x, \quad x \in [0, 2].$

Solution: Factor: $y = x(x - 1)(x - 2)$, with zeros at $x = 0, 1, 2$. On $(0, 1)$ the factors give $y > 0$; on $(1, 2)$, $y < 0$. The area is

$$\int_0^1 (x^3 - 3x^2 + 2x) dx - \int_1^2 (x^3 - 3x^2 + 2x) dx.$$

With antiderivative $\frac{x^4}{4} - x^3 + x^2$, the first integral is $\frac{1}{4} - 1 + 1 = \frac{1}{4}$, and the second is $(4 - 8 + 4) - \frac{1}{4} = -\frac{1}{4}$. Hence the area is

$$\frac{1}{4} - \left(-\frac{1}{4} \right) = \frac{1}{2}.$$

(b) $y = x\sqrt{4 - x}, \quad x \in [0, 4].$

Solution: On $[0, 4]$, $x \geq 0$ and $\sqrt{4 - x} \geq 0$, so $y \geq 0$ and the area is $\int_0^4 x\sqrt{4 - x} dx$. Let $u = 4 - x$, so $x = 4 - u$ and $dx = -du$; when $x = 0$, $u = 4$ and when $x = 4$, $u = 0$. Then

$$\int_0^4 x\sqrt{4 - x} dx = \int_4^0 (4 - u)\sqrt{u}(-du) = \int_0^4 (4 - u)u^{1/2} du = \int_0^4 (4u^{1/2} - u^{3/2}) du.$$

Evaluating,

$$= \left[\frac{8}{3}u^{3/2} - \frac{2}{5}u^{5/2} \right]_0^4 = \frac{8}{3}(8) - \frac{2}{5}(32) = \frac{64}{3} - \frac{64}{5} = \frac{128}{15}.$$