Seminar Talk: Excess Intersection Formula

Chapter 13 of <u>3264 and all that</u>.

Dec 20

§1 Five conics revisited

<u>Recall</u> (3264 Problem)

Given 5 general conics $C_1, \cdots, C_5 \subseteq \mathbb{P}^2 / \mathbb{C}$,

how many (smooth) conics are tangent to all of them?

Answer: $\{\text{plane conics}\} \cong \mathbb{P}^5$

$\rightsquigarrow Z_i = \{\text{plane conics tangent to } C_i\} \subseteq \mathbb{P}^5$

can be realized as a hypersurface of deg 6.

Attempt: $|Z_1 \cap \cdots \cap Z_5| \neq$ Euler number $(\mathcal{O}_{\mathbb{P}^5}(6)^{\oplus 5}) = 6^5 = 7776$.

However, this is the wrong answer b/c

the "excess intersections" lies in it.

$$\begin{tikzcd} & & \bigcap_{i=1}^{5} Z_i \arrow[d, phantom, "\cap"] \\ \text{linear poly's} & \mathbb{P}^2 \arrow[ur, hook] \arrow[r, hook, "\text{Veronese}"', "\ell \mapsto \ell^2"] & \mathbb{P}^5 \end{tikzcd}$$

Once we remove $\mathbb{P}^2$

$\rightsquigarrow \bigcap Z_i \cong \underbrace{\mathbb{P}^2}_{\text{excess part}} \cup \underbrace{\Gamma}_{\text{finite part}}$ set-theoretically

"Excess": doesn't have a proper dim for the intersection.

with $\deg \Gamma = 7776 - 4512 = 3264$

The number "4512" comes from the application of Excess Intersection Formula.

§2 Excess Intersection Formula

<u>Theorem</u> X sm var, $S \subseteq X$ subvar, $T \subseteq X$ lci, then

$$[S]\cdot[T] := \sum_C (\iota_C)_*(\gamma_C).$$

where ① sum taken over the connected components C of $S\cap T$

② $\iota_C : C \hookrightarrow X$ inclusion morphism

③ $\gamma_C = \left\{ \frac{c(N_{T/X}|_C)}{c(N_{C/S})} \right\}_d^{\,a} \in A_d(C)$ the codim d strata

with d = excepted dim of $S\cap T$

$= \dim X - \operatorname{codim} S - \operatorname{codim} T$.

Remarks (1) lci condition is necessary whereas smooth is not.

(RHS doesn't require smooth condition).

· can define intersection of arbitrary lci subvarieties.

· can replace lci by "smooth".

Details: $T_1, T_2 \subseteq Y$ arbitrary, Y sm

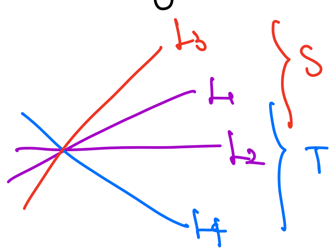
$\Rightarrow Y \cong \Delta_Y \subseteq Y\times Y$ automatically lci,

& $\Delta_Y \cap (T_1\times T_2) \cong T_1\cap T_2$ lci.

(2) Irred comps $\underset{?}{\overset{\text{dividing}}{\rightsquigarrow}}$ conn. comps

No canonical way!

E.g. $L_1, \cdots, L_4 \subseteq \mathbb{P}^2$, S & T cubic curves



s.t. $[S][T] \in A(\mathbb{P}^2)$, $\deg [S][T] = 9$

no canonical way to divide

9 = sum of two components

The Geometric Construction of γ_C

X sm proj var of dim n. $S, T \subset X$ with codim k, l, resp.

$\rightsquigarrow C = S\cap T = \coprod_\alpha C_\alpha$, $\operatorname{codim} C_\alpha = k + l - \underset{\smile}{m_\alpha}$.

Outline: $\gamma_\alpha \in A^{m_\alpha}(C_\alpha)$ the excess part.

the excess part is "truncated" so that

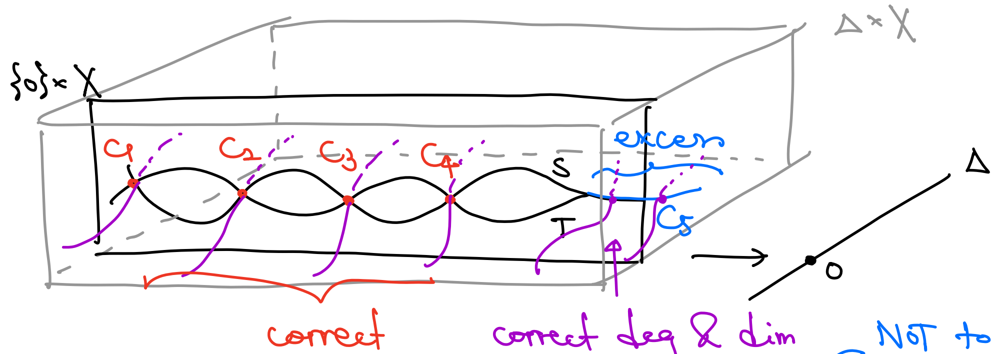
$$\dim = \dim C_\alpha - m_\alpha = (n - \operatorname{codim} C_\alpha) - m_\alpha = n-k-l$$

as excepted.

$$\rightsquigarrow \quad \sum_\alpha (i_\alpha)_*(\gamma_\alpha) = [S]\cdot[T] \in A^{k+l}(X).$$

Apply deformation. Flat families $\mathcal{S}, \mathcal{T} / \Delta$ = sm rat curve.

s.t. $S_0 = S$, $T_0 = T$. $S_\lambda \cap T_\lambda$ transverse when $\lambda \neq 0$.



$$\mathcal{S} \cap \mathcal{T} \begin{cases} \text{transverse } \Delta \setminus \{0\} \quad \text{(unimportant)} \\ \quad \Phi|_X = \Phi \setminus \{0\}\times X \\ \text{singular } \{0\} \in \Delta \begin{cases} \text{excess part} \\ \text{correct part } \ \Sigma \overset{\overline{(\Delta\setminus\{0\})\times X}}{=} \Phi \cap (\{0\}\times X) \end{cases} \\ \quad D = \{0\}\times C \end{cases}$$

NOT to throw it away, but to "cut out" some dim.

$\dim = n-k-l$

Now, $\boxed{\Sigma = \coprod_\alpha \Sigma_\alpha = \coprod_\alpha \Phi \cap (\{0\}\times C_\alpha)}$.

note: if $\exists p \in \{0\}\times C_\alpha$ s.t. $T_p(\{0\}\times C_\alpha) \neq T_p\mathcal{S} \cap T_p\mathcal{T}$,

namely $\{0\}\times C_\alpha$ lies in the excess part,

then $\Phi \cap (\{0\}\times C_\alpha)$ never goes to be excess.

$\Rightarrow$ It is safe to run through all α.

Consider $\mathcal{N}_{\{0\}\times C_\alpha / \Delta\times X} \longrightarrow \mathcal{N}_{\mathcal{S}/\Delta\times X}|_{\{0\}\times C_\alpha} \oplus \mathcal{N}_{\mathcal{T}/\Delta\times X}|_{\{0\}\times C_\alpha}$

rank of bundles $\quad (n+1)-(n-k-l+m_\alpha) \qquad (n+1)-(n-k+1) \ + \ (n+1)-(n-l+1)$

on $\{0\}\times C_\alpha \quad = k+l-m_\alpha+1 \qquad = \quad k+l$.

It fails to be injective on $(\{0\}\times C_\alpha)\cap\Phi$ for $m_\alpha = 0$.

Now, $c(\mathcal{N}_{\{0\}\times C_\alpha/\Delta\times X}) = c(\mathcal{N}_{C_\alpha/X})$,

$c(\mathcal{N}_{\mathcal{S}/\Delta\times X}|_{\{0\}\times C_\alpha}) = c(\mathcal{N}_{S/X}|_{C_\alpha})$, $c(\mathcal{N}_{\mathcal{T}/\Delta\times X}|_{\{0\}\times C_\alpha}) = c(\mathcal{N}_{T/X}|_{C_\alpha})$.

Recall Porteous Formula:

$\operatorname{codim} M_k(\varphi) = (e-k)(f-k)$, $\varphi: \mathcal{E} \to \mathcal{F}$ b/w v.b.

with $\operatorname{rank}\mathcal{E} = e$, $\operatorname{rank}\mathcal{F} = f$

$$\Rightarrow [M_k(\varphi)] = \left\{\frac{c(\mathcal{F})}{c(\mathcal{E})}\right\}^{f-e+1} \cdot \left[\frac{c(\mathcal{F})}{c(\mathcal{E})}\right].$$

Apply Porteous to $\Sigma = \Phi \cap (\{0\}\times C_\alpha)$:

$$\gamma_\alpha = [\Sigma_\alpha] = \boxed{\left\{\frac{c(\mathcal{N}_{S/X}|_{C_\alpha})\cdot c(\mathcal{N}_{T/X}|_{C_\alpha})}{c(\mathcal{N}_{C_\alpha/X})}\right\}^{m_\alpha}} \in A^{m_\alpha}(\{0\}\times C_\alpha)$$

check: $e = \operatorname{rank}\mathcal{E} = k+l-m_\alpha+1$. $f = \operatorname{rank}\mathcal{F} = k+l$

$(e-k)(f-k) = (e-(k+l+1))(f-(k+l+1)) = m_\alpha$

$= \dim(\{0\}\times C_\alpha) - \dim \Sigma_\alpha$

$\rightsquigarrow f-e+1 = m_\alpha$.

Finally, when S is lci,

$0 \to \mathcal{N}_{C/S} \to \mathcal{N}_{C/X} \to \mathcal{N}_{S/X}|_C \to 0$ exact.

By Whitney: $c(\mathcal{N}_{C/X}) = c(\mathcal{N}_{C/S})\cdot c(\mathcal{N}_{S/X}|_C)$.

$\Rightarrow \gamma_\alpha = \{c^{-1}(\mathcal{N}_{C_\alpha/S})\cdot c(\mathcal{N}_{T/X}|_{C_\alpha})\}^{m_\alpha}$ as in EIF.

Final Remark
- 2_* does not change γ_α when $m_\alpha = 0$.
- 2_* realizes $(\{0\}\times C_\alpha)\cap\Phi$ as $C_\alpha \subseteq X$ when $m_\alpha > 0$.

§3 Applications

Keynote Question 1 $S_1, S_2, S_3 \subseteq \mathbb{P}^3$ surfaces of deg $= s_1, s_2, s_3$.

And $S_1 \cap S_2 \cap S_3 = C \sqcup \Gamma$, $C = (d,g)$-sm curve. What is $\deg\Gamma$?

Solution. By EIF: $m_\Gamma = 0$, $m_C = 1$, $X = \mathbb{P}^3$.

$$[S_1]\cdot[S_2]\cdot[S_3] = 2_*\{c^{-1}(\mathcal{N}_{T/S_3})\cdot c(\mathcal{N}_{S_1/X}|_T)\cdot c(\mathcal{N}_{S_2/X}|_T)\}_0$$

$$+\ 2_*\{\underbrace{c^{-1}(\mathcal{N}_{C/S_3})}_{c(\mathcal{N}_{S_3/X}|_C)\cdot c^{-1}(\mathcal{N}_{C/X})}\cdot c(\mathcal{N}_{S_1/X}|_C)\cdot c(\mathcal{N}_{S_2/X}|_C)\}_1$$

$$\Rightarrow \deg [S_1]\cdot[S_2]\cdot[S_3] = s_1 s_2 s_3$$

$$= \deg \Gamma - \underbrace{\deg c_1(\mathcal{N}_{C/\mathbb{P}^3})} + \sum \overbrace{\deg c_1(\mathcal{N}_{S_i/\mathbb{P}^3}|_C)}^{ds_i}$$

$= \deg c_1(\mathcal{N}_{C/S_i}) + \deg c_1(\mathcal{N}_{S_i/\mathbb{P}^3}|_C)$ for some S_i

$$= \underline{\deg [C_{S_i}]^2} + ds_i$$

$= 2g-2-ds_i+4d = (\deg$ of $[C]^2 \in A(S_i))$

(*) By adjunction formula:

$$\deg K_C = 2g-2 = \deg K_{S_i}|_C + \deg [C_{S_i}]^2$$

hypersurface class $\in A^1(S_2)$

Here $K_{S_i} = c_1(\mathcal{N}_{S_i/\mathbb{P}^3}) + K_{\mathbb{P}^3}|_{S_i} = c_1(\mathcal{N}_{S_i/\mathbb{P}^3}) + 4[H]$

$n+1$, see chapter 7

$$\Rightarrow K_{S_i}|_C = c_1(\mathcal{N}_{S_i/\mathbb{P}^3}|_C) + 4[H]|_C$$

$$\Rightarrow \deg c_1(\mathcal{N}_{S_i/\mathbb{P}^3}|_C) = 2g-2-\deg[C_{S_i}]^2 + 4d = ds_i$$

$$\Rightarrow \deg [C_{S_i}]^2 = 2g-2+4d-ds_i$$

$\Rightarrow s_1 s_2 s_3 = \deg \Gamma + 2g-2+4d+d(s_1+s_2+s_3)$ as our wish.

<u>Keynote Question 2</u> $S, T \subseteq \mathbb{P}^4$ of deg s & t. $S\cap T = C \sqcup \Gamma$, where $C = (d,g)$-sm curve. What is $\deg\Gamma$?

<u>Solution</u>. By EIF: $st = \deg\Gamma + \deg\{c^{-1}(\mathcal{N}_{C/S})\cdot c(\mathcal{N}_{T/\mathbb{P}^4}|_C)\}_0$

$$= \deg\Gamma - \deg c_1(\mathcal{N}_{C/S}) + \deg c_1(\mathcal{N}_{T/\mathbb{P}^4}|_C)$$

Repeat similar argument above.

$$\Rightarrow \deg\Gamma = st - 2g + 2 - 5d + \underbrace{\deg[C_S]^2 + \deg[C_T]^2}_{}$$

need more conditions to get it explicitly

Back to 5 conics

$\{\text{plane conics}\} \cong \mathbb{P}^5$

$\rightsquigarrow Z_i = \{\text{plane conics tangent to } C_i\} \subseteq \mathbb{P}^5$, $\quad \bigcap Z_i = S \amalg \Gamma$, $\quad S \cong \mathbb{P}^2$

can be realized as a hypersurface of deg 6.

Let $\zeta = [L] \in A^1(S)$, $\eta = [H] \in A^1(\mathbb{P}^5)$, $\eta|_S = 2\zeta$ by Veronese. $\zeta^3 = 0$.

Note: $\mathcal{N}_{Z_i/\mathbb{P}^5} = \mathcal{O}_{Z_i}(6)$, $c(\mathcal{N}_{Z_i/\mathbb{P}^5}|_S) = 1 + 12\zeta$.

- $0 \to \mathcal{O}_{\mathbb{P}^2} \to \mathcal{O}_{\mathbb{P}^2}(1)^3 \to T_{\mathbb{P}^2} \to 0$

  $\Rightarrow \quad c(T_S) = (1+\zeta)^3 = 1 + 3\zeta + 5\zeta^2$

- $0 \to \mathcal{O}_{\mathbb{P}^5} \to \mathcal{O}_{\mathbb{P}^5}(1)^6 \to T_{\mathbb{P}^5} \to 0$

  $\Rightarrow \quad c(T_{\mathbb{P}^5}|_S) = (1+\eta)^6|_S = (1+2\zeta)^6 = 1 + 12\zeta + 60\zeta^2$

- $0 \to T_S \to T_{\mathbb{P}^5}|_S \to \mathcal{N}_{S/\mathbb{P}^5} \to 0$

  Even if $\mathcal{N}_{S/\mathbb{P}^5}$ cannot be embeded back,

  Whitney says $c(\mathcal{N}_{S/\mathbb{P}^5}) = \dfrac{1 + 12\zeta + 60\zeta^2}{1 + 3\zeta + 5\zeta^2} = 1 + 9\zeta + 30\zeta^2$.

  $\Rightarrow c^{-1}(\mathcal{N}_{S/\mathbb{P}^5}) = 1 - 9\zeta + 51\zeta^2$.

Fact: Scheme-theoretically $\bigcap Z_i = T \amalg \Gamma = V(\mathcal{I}^2_{S/\mathbb{P}^5}) \amalg \Gamma$

This is precisely the push-forward step.

$\text{mult}_S(Z_i) = 2$ by Riemann-Hurwitz

$(c^{-1})_k(\mathcal{N}_{T/\mathbb{P}^5}) = 2^{k+3} \cdot (c^{-1})_k(\mathcal{N}_{S/\mathbb{P}^5})$

by definition of Segre classes

$\Rightarrow c^{-1}(\mathcal{N}_{T/\mathbb{P}^5}) = 2^3 \cdot 1 - 2^4 \cdot 9\zeta + 2^5 \cdot 51\zeta^2$

$= 8 - 144\zeta + 1632\zeta^2$

Hence $\deg\left(c^{-1}(\mathcal{N}_{T/\mathbb{P}^5}) \cdot \prod_{i=1}^{5} c(\mathcal{N}_{Z_i/\mathbb{P}^5}|_S)\right)$

$= \deg\left((1+12\zeta)^5 \cdot (8 - 144\zeta + 1632\zeta^2)\right)$ taking the coeff of ζ^2.

$= 1632 - 144 \cdot 12 \cdot 5 - 8 \cdot 12^2 \cdot 10 = 4512$

$\Rightarrow \deg \Gamma = 6^5 - 4512 = 3264$.

Thanks! :)