Problem Set

Your solutions need to be typed by LaTeX or written neatly and scanned properly. You must submit them via Tsinghua University's Web Learning by 11:59 pm on Tuesday, October 31. *No late submission will be accepted.*

You are encouraged to discuss the problems with your classmates, but you must write your solutions on your own and acknowledge collaborators/cite references if any. Failure to do so will be treated as academic dishonesty.

1. For each of the following, give one example and explains briefly why your example works.
   (a) A local ring A such that its maximal ideal is generated by a non-nilpotent element but A is not a discrete valuation ring.
   (b) A finite separable extension L/K of complete discrete valuation fields whose residue field extension k_L/k is not separable.

2. Let K be a field. A *non-trivial non-archimedean absolute value* on K is a function $|\cdot|\colon K \to \mathbb{R}_{\geq 0}$ satisfying for $x, y \in K$: (i) $|xy| = |x| \cdot |y|$; (ii) $|x+y| \leq \max\{|x|, |y|\}$; (iii) $|x| = 0$ if and only if $x = 0$; (iv) $|K| \supsetneq \{0, 1\}$. An absolute value defines a topology on K in a usual way.

   Now let $|\cdot|_1$ and $|\cdot|_2$ be two non-trivial non-archimedean absolute values on K. Show that they give the same topology if and only if there exists $\rho > 0$ such that $|x|_2 = |x|_1^{\rho}$ for every $x \in K$.

3. Let K be a complete discrete valuation field with valuation v and let L/K be a finite field extension of degree n. Then we showed that L admits a unique valuation w such that $w|_K = v$ (here we normalize so that w prolongs v with index 1, not index $e_{L/K}$).

   This exercise outlines another proof of this result by an explicit formula. Define $w\colon L \to \mathbb{R} \cup \{\infty\}$ by
   $$w(x) = \frac{1}{n} v\big(N_{L/K}(x)\big) \quad (x \in L).$$
   It is easy to see w is non-trivial, $w|_K = v$, and $w(xy) = w(x) + w(y)$. We are going to show
   $$w(x+y) \geq \min\big\{w(x), w(y)\big\} \quad \text{for} \quad x, y \in L.$$
   Note that the uniqueness of the prolonged norm follows from the property of topological vector spaces as we saw in the class.
   (a) Show that it suffices to prove, for $x \in L$, $w(x) \geq 0$ implies $w(x+1) \geq 0$.
   (b) Take any $x \in L$ with $w(x) \geq 0$. Show $w(x+1) \geq 0$.
   Hint: let $f(X) = X^m + a_{m-1}X^{m-1} + \cdots + a_1 X + a_0 \in K[X]$ be the minimal polynomial of x (so $m \mid n$). Express $w(x)$ in terms of a_m and n. Similarly, using the fact that $f(X-1)$ is the minimal polynomial of $x+1$, express $w(x+1)$ in terms of a_i's.

4. [Ser79, p. 53, Exercise].
5. [Ser79, p. 71, Exercise 2]. You may use the result of Exercise 1 without proof.
6. [Ser79, p. 72, Exercise 5].
7. [Ser79, p. 116, Exercise].
8. [Ser79, p. 119, Exercise 1].

References

[Ser79] Jean-Pierre Serre, *Local fields*, Graduate Texts in Mathematics, vol. 67, Springer-Verlag, New York-Berlin, 1979, Translated from the French by Marvin Jay Greenberg.