Problem Set

Your solutions need to be written *in English* and either typed by LaTeX or written neatly and scanned properly. You must submit them via Tsinghua University's Web Learning by 11:59 pm on Tuesday, December 19. *No late submission will be accepted.*

You are encouraged to discuss the problems with your classmates, but you must write your solutions on your own and acknowledge collaborators/cite references if any. Failure to do so will be treated as academic dishonesty.

1. Prove that the multiplicative group $K^\times$ of the non-archimedean local field $K = \mathbb{F}_p(\!(t)\!)$ has a non-closed subgroup of finite index.

2. Let K be a non-archimedean local field with $\operatorname{char} K \neq 2$ and let $(-,-)_v \colon K^\times \times K^\times \to \{\pm 1\}$ denote the local symbol defined in the class and [Ser79, p. 208] for $n = 2$. Show that for each $a, b \in K^\times$, $(a,b)_v = 1$ if and only if there exists $x, y, z \in K$ such that $z^2 = ax^2 + by^2$.[1] Hint: Use [Ser79, p. 208, Prop. 7 iii)].

3. Let $p \geq 3$. For each $n \geq 1$, let $\mu_n := \{\zeta \in \overline{\mathbb{Q}}_p \mid \zeta^n = 1\}$.
   (a) Show $\mu_{p-1} \subset \mathbb{Q}_p$.
   (b) Show $\mathbb{Q}_p(\mu_p) = \mathbb{Q}_p(\sqrt[p-1]{-p})$, where $\sqrt[p-1]{-p}$ denotes a root of $x^{p-1} + p = 0$ in $\overline{\mathbb{Q}}_p$.
   (c) Consider the following isomorphisms

$$\overline{\sigma} \colon (\mathbb{Z}/p\mathbb{Z})^\times \xrightarrow{\cong} \operatorname{Gal}(\mathbb{Q}_p(\mu_p)/\mathbb{Q}_p); \quad a \mapsto (\overline{\sigma}_a \colon \zeta_p \mapsto \zeta_p^a) \quad (\zeta_p \in \mu_p),$$

   and

$$\theta_0 \colon \operatorname{Gal}(\mathbb{Q}_p(\mu_p)/\mathbb{Q}_p) \xrightarrow{\cong} (\mathbb{Z}/p\mathbb{Z})^\times; \quad g \mapsto g(\pi)/\pi \quad (\pi \in \mathbb{Z}_p[\mu_p] \quad \text{is a uniformizer}).$$

   Here the second map θ_0 is defined in [Ser79, p. 67, Prop. 7] and is an isomorphism since $\mathbb{Q}_p(\mu_p)/\mathbb{Q}_p$ is a tamely ramified extension of degree $p-1$. Show $\theta_0 \circ \overline{\sigma} = \mathrm{id}$.

4. Keep the assumption and notation as in Problem 3. Consider the local Artin map (reciprocity map)

$$\operatorname{Art}_p = (\,,{}^*/\mathbb{Q}_p) \colon \mathbb{Q}_p^\times \to \operatorname{Gal}(\mathbb{Q}_p^{\mathrm{ab}}/\mathbb{Q}_p)$$

   with the arithmetic normalization as in [Ser79]. Write $\mathbb{Q}_p(\mu_{p^\infty}) := \bigcup_{m \geq 1} \mathbb{Q}_p(\mu_{p^m})$ and fix the identification

$$\mathbb{Z}_p^\times \xrightarrow{\cong} \operatorname{Gal}(\mathbb{Q}_p(\mu_{p^\infty})/\mathbb{Q}_p); \quad a \mapsto \left(\sigma_a \colon \zeta_{p^m} \mapsto \zeta_{p^m}^{a \bmod p^m}\right).$$

   Let $u \in \mathbb{Z}_p^\times$ be a primitive $(p-1)$st root of unity (which exists by Problem 3(a)). We are going to show $\operatorname{Art}_p(u)|_{\mathbb{Q}_p(\mu_{p^\infty})} = \sigma_{u^{-1}}$.

[1] This holds in a more general setup if we use the symbol $(-,-)$ instead (see [Ser79, p. 207, Remark 3)]).

(a) Let $(-,-)_v\colon \mathbb{Q}_p^\times \times \mathbb{Q}_p^\times \to \mu_{p-1}$ denote the local symbol defined in the class and [Ser79, p. 208] for $n = p-1$. Show $(u,-p)_v = u$.

(b) Deduce $\mathrm{Art}_p(u)|_{\mathbb{Q}_p(\mu_p)} = \overline{\sigma}_{u^{-1}}$.

(c) Show $\mathrm{Art}_p(u)|_{\mathbb{Q}_p(\mu_{p^\infty})} = \sigma_{u^{-1}}$.

5. Let $K = \mathbb{F}_p(t)$ and let $\mathbb{A}_K$ denote its adèle ring. Show that K is discrete in $\mathbb{A}_K$ and the quotient $\mathbb{A}_K/K$ is compact (with respect to the quotient topology).

6. Let K be a global field and let $\mathbb{I}_K$ denote its idèle group. Show that the inverse map $\mathbb{I}_K \to \mathbb{I}_K; x \mapsto x^{-1}$ is not continuous if $\mathbb{I}_K$ is equipped with the induced topology $\mathbb{I}_K \subset \mathbb{A}_K$ from the adèle ring.

7. Recall that $K^\times$ embeds into $\mathbb{I}_K$ diagonally for every global field K.

   (a) Show that $\mathbb{Q}^\times$ and $\prod_p \mathbb{Z}_p^\times \times \mathbb{R}_{>0}$ generate $\mathbb{I}_\mathbb{Q}$, and $\mathbb{Q}^\times \cap \left(\prod_p \mathbb{Z}_p^\times \times \mathbb{R}_{>0}\right) = \{1\}$.

   (b) Let $K = \mathbb{Q}(\sqrt{-5})$. Show that $\mathbb{I}_K$ is not generated by $K^\times$ and $\prod_{v \in S_{K,\mathrm{fin}}} \mathcal{O}_{K_v}^\times \times \mathbb{C}^\times$.

8. For $n \geq 1$, let $\mathbb{Q}(\mu_n)$ denote the cyclotomic field generated by nth roots of unity and let $N\colon \mathbb{I}_{\mathbb{Q}(\mu_n)} \to \mathbb{I}_\mathbb{Q}$ be the norm map. Construct explicitly a group isomorphism

$$\mathbb{I}_\mathbb{Q}/(\mathbb{Q}^\times N(\mathbb{I}_{\mathbb{Q}(\mu_n)})) \xrightarrow{\cong} (\mathbb{Z}/n\mathbb{Z})^\times.$$

   Moreover, describe the image in $(\mathbb{Z}/n\mathbb{Z})^\times$ of the following idèles:

   (a) $\pi_p = (1,\ldots,1,p,1,\ldots,1)$ (p sits in the $\mathbb{Q}_p$-component) for $(p,n)=1$;

   (b) $c = (1,1,\ldots,-1)$ (-1 sits in the $\mathbb{R}$-component and the other entries are 1).

   You may use any result on the image of the local norm map $N_{\mathbb{Q}_p(\mu_n)/\mathbb{Q}_p}\colon \mathbb{Q}_p(\mu_n) \to \mathbb{Q}_p$ as long as you state it correctly.

References

[Ser79] Jean-Pierre Serre, *Local fields*, Graduate Texts in Mathematics, vol. 67, Springer-Verlag, New York-Berlin, 1979, Translated from the French by Marvin Jay Greenberg.