COURSEWORK FOR ALGEBRAIC TOPOLOGY I (SPRING 2025)

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This document is about the course *Algebraic Topology I* offered by Qiuzhen College, Tsinghua University, during the Spring 2025 semester. The following contains 8 sheets of homework problems, a suggested list of additional exercise problems (without solutions), and the (closed-book) final exam. All problems are proposed by the lecturer and are attached with solutions by the TA who is responsible for any mistakes in this document.

Contents

Homework 1

Notations. Given a space X, we let

$$Z_n(X) \coloneqq \ker(d\colon S_n(X) \to S_{n-1}(X)),$$
$$B_n(X) \coloneqq \operatorname{im}(d\colon S_{n+1}(X) \to S_n(X)),$$

to be the n-th cycle group and n-th boundary group of X, respectively.

Problem 1.1. Let X be a space. Solve the following problems.

(1) Given two paths $\sigma_1, \sigma_2\colon \Delta^1 \to X$ with the same start and terminal points. Suppose σ_1 and σ_2 are homotopic relative to boundary. Show that $\sigma_1 - \sigma_2 \in B_1(X)$.
(2) Show that there is a homomorphism $\pi_1(X) \to H_1(X)$ that sends each loop to its homology class.
(3) Suppose X is path-connected, show that this homomorphism is surjective.

Solution. (1) Since σ_1 and σ_2 are homotopic relative to boundary, there exists a homotopy map $H\colon \Delta^1 \times [0,1] \to X$ such that $H(t,0) = \sigma_1(t)$ and $H(t,1) = \sigma_2(t)$ for all $t \in \Delta^1$, as well as $H(0,s) = H(1,s)$ for all $s \in [0,1]$.

Identify $\Delta^1 \times [0,1]$ with the square $[0,1]^2$ and divide it into two disjoint triangles $[0,1]^2 = T_1 \sqcup T_2$ along the diagonal. Denote e_{ij} the j-th boundary edge of T_i for $i = 1,2$ and $j = 0,1,2$. Now it is clear that T_1, T_2 are homeomorphic to Δ^2. By properly choosing the homeomorphisms, we require the following correspondence between edges e_{ij} of T_1, T_2 and edges on boundary of Δ^2:

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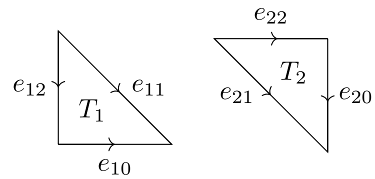


Then, using $d_j H|_{T_i} = e_{ij}$ we can compute

$$\begin{aligned} d(H|_{T_1} - H|_{T_2}) &= e_{10} - e_{11} + e_{12} - e_{20} + e_{21} - e_{22} \\ &= (e_{10} - e_{22}) + (e_{12} - e_{20}) \\ &= \sigma_1 - \sigma_2 + (e_{12} - e_{20}). \end{aligned}$$

This is clearly a 1-boundary. Since $H(0, s) = H(1, s)$, the term $e_{12} - e_{20} \in B_1(X)$ is a linear combination of two points, and hence lies in $B_1(X)$. So we have $\sigma_1 - \sigma_2 \in B_1(X)$ as desired.

(2) For each $\gamma \in \pi_1(X)$, let $[\gamma]$ be its homology class modulo $B_1(X)$. Then the map $\pi_1(X) \to H_1(X)$, $\gamma \mapsto [\gamma]$ is well-defined for the following reason: If loops γ, γ' define the same element in $\pi_1(X)$, then they are homotopic, and hence $\gamma - \gamma' \in B_1(X)$ by (1), which means $[\gamma] = [\gamma']$. Moreover, this map is a homomorphism because $[\gamma_1] + [\gamma_2] = [\gamma_1 \circ \gamma_2]$ for $\gamma_1, \gamma_2 \in \pi_1(X)$ inside $Z_1(X)$.

(3) Fix $x_0 \in X$ as the basepoint of $\pi_1(X)$. We aim to find the preimage of an arbitrary $[\sigma] \in H_1(X)$. Since $\pi_1(X) \to H_1(X)$ is a homomorphism by (2), we reduce to the case where σ is homotopic to a single loop with some basepoint $x_\sigma \in X$. Applying the path-connectedness, there exists a path p starting at x_0 and ending at x_σ. Then $p^{-1} \circ \sigma \circ p$ and σ are homotopic, so we obtain $[p^{-1} \circ \sigma \circ p] = [\sigma]$ in $H_1(X)$ by (1). In this case, $p^{-1} \circ \sigma \circ p \in \pi_1(X)$ with basepoint x_0 is the preimage of $[\sigma]$ along $\pi_1(X) \to H_1(X)$. This completes the proof of surjectivity. □

Problem 1.2. Let $f : \mathbb{R} \to S^1$ be the universal cover $f(\theta) = e^{2\pi i\theta}$. Given a path $\sigma : \Delta^1 \to S^1$, we pick any lift $\tilde{\sigma} : \Delta^1 \to \mathbb{R}$ of σ and define the winding number of σ as $w(\sigma) := \tilde{\sigma}(e_1) - \tilde{\sigma}(e_0) \in \mathbb{R}$. Given a singular chain $c = \sum_{i=1}^k a_i \sigma_i \in C_1(S^1)$, we define its winding number as $w(c) = \sum_{i=1}^k a_i w(\sigma_i)$.

(1) Show that $w(c) \in \mathbb{Z}$ if $c \in Z_1(S^1)$.
(2) Show that $w(c) = 0$ if $c \in B_1(S^1)$.
(3) Show that w induces an isomorphism $H_1(S^1) \cong \mathbb{Z}$.

Solution. One can check that the winding number w is well-defined, i.e. $w(\sigma)$ is independent of the choice of $\tilde{\sigma}$.

(1) By definition, if $c \in Z_1(S^1)$ then $d(c) = d_0(c) - d_1(c) \in S_0(X)$ must be trivial, in which case the start and terminal points of c are the same. On the other hand, note that the universal cover f induces $S^1 \simeq \mathbb{R}/\mathbb{Z}$. Thus, through σ the start and terminal points of c have the same image in $S^1 \simeq \mathbb{R}/\mathbb{Z}$, and hence their images are only differed by $\mathbb{Z}$ through the lift $\tilde{\sigma}$. It follows that $w(\sigma) \in \mathbb{Z}$.

(2) By definition, if $c \in B_1(S^1)$ then $c = d(\tau) : \Delta^1 \to S^1$ for some 2-simplex $\tau : \Delta^2 \to S^1$. Regarding c as a path, pick a lift $\tilde{c} : \Delta^1 \to \mathbb{R}$ by picking a lift $\tilde{\tau} : \Delta^2 \to \mathbb{R}$ such that $\tilde{c} = d(\tilde{\tau})$. As boundary of a 2-simplex, $\tilde{c}$ must be a closed loop in $\mathbb{R}$ up to homotopy, and hence $\tilde{c}(e_1) = \tilde{c}(e_0)$. This shows $w(c) = 0$ for all $c \in B_1(S^1)$.

(3) For any class $[\sigma] \in H_1(S^1)$, let $\sigma \in Z_1(S^1)$ be its representative; then we have $w(\sigma) \in \mathbb{Z}$ by (1). This defines the map

$$\varphi : H_1(S^1) \longrightarrow \mathbb{Z}, \quad [\sigma] \longmapsto w(\sigma).$$

From (2), we see this map is well-defined. We show that φ is an isomorphism of groups as follows. Given σ_i with boundary points e_{i0}, e_{i1} for $i = 1, 2$, whenever $e_{10} = e_{21}$, we observe that $\tilde{\sigma}(e_{11}) - \tilde{\sigma}(e_{20}) = \tilde{\sigma}(e_{11}) - \tilde{\sigma}(e_{10}) + \tilde{\sigma}(e_{21}) - \tilde{\sigma}(e_{20})$, or equivalently $w(\sigma_2 \circ \sigma_1) = w(\sigma_1) + w(\sigma_2)$. On the other hand, Problem 1.1(2) implies $[\sigma_2 \circ \sigma_1] = [\sigma_1] + [\sigma_2]$. So φ is a group homomorphism. Next, as φ is a homomorphism, for the surjectivity of φ, it suffices to find an element in $H_1(S^1)$ with winding number $1 \in \mathbb{Z}$ (notice that any $c \in B_1(S^1)$ gives $w(c) = 0$ by (2)). But this is easy as such an element can come from $\pi_1(S^1)$ by Problem 1.1(3). Finally, the injectivity of φ is implied by the converse of part

(2), which can be proved by simply converting the argument before. Therefore, we conclude that φ is an isomorphism of groups. $\square$

Problem 1.3 (MIT 18.905, Problem Set I, Problem 4)**.**

> Here are a couple more "categorical" definitions, giving you some practice with the idea of constructions being defined by universal mapping properties.

Let $\mathcal{C}$ be a category, A a set, and $a \mapsto X_a$ an assignment of an object of $\mathcal{C}$ to each element of A. A *product* of these objects in an object Y together with maps $\mathrm{pr}_a : Y \to X_a$ with the following property. For any object Z and any family of maps $f_a : Z \to X_a$, there is a unique map $f : Z \to Y$ such that $f_a = \mathrm{pr}_a \circ f$ for all $a \in A$. A *coproduct* of these objects is an object Y together with maps $\mathrm{in}_a : X_a \to Y$ with the following property. For any object Z and any family of maps $f_a : X_a \to Z$, there is a unique map $f : Y \to Z$ such that $f_a = f \circ \mathrm{in}_a$ for all $a \in A$.

(1) Describe constructions of the product and coproduct (if they exist) in the following categories: sets, pointed sets, spaces, abelian groups. (A pointed set is a pair $(S, *)$ where S is a set and $* \in S$.)
(2) What should be meant by the product when $A = \emptyset$? How about the coproduct? What are these objects in the four categories mentioned in (1)? Give an example of a category in which neither one of these constructions exists.
(3) Show that if $(Y, \{\mathrm{pr}_a\})$ and $(Y', \{\mathrm{pr}'_a\})$ are both products of a family $\{X_a : a \in A\}$, then there is a unique map $f : Y \to Y'$ such that $\mathrm{pr}'_a \circ f = \mathrm{pr}_a$ for all $a \in A$, and that this map is an isomorphism.
(4) A partially ordered set S defines a category as follows. The objects are the elements of S. (They constitute a *set*, rather than something larger, so this is a "small category".) For $s, s' \in S$, there is exactly one morphism $s \to s'$ if $s \leqslant s'$, and none otherwise. Take S to be the real numbers $\mathbb{R}$ with their natural order, for example, and consider a map $A \to \mathbb{R}$. Under what conditions does the product of these objects exist, and if it does what is it? Same question for the coproduct.

Solution. (1) All products and coproducts can be constructed directly by definition. The results are listed as follows.

- In sets, the product is the Cartesian product $\prod_{a \in A} X_a$, and the coproduct is the disjoint union $\coprod_{a \in A} X_a$.
- In pointed sets, the product is the Cartesian product with the pair of basepoints $\prod_{a \in A}(X_a, *_a)$, and the coproduct is the wedge sum $\bigvee_{a \in A}(X_a, *_a)$, which is a quotient of disjoint union formed by identifying the basepoints of all X_a's.
- In spaces, the product is $\prod_{a \in A} X_a$ with the product topology, and the coproduct is $\coprod_{a \in A} X_a$ with the disjoint union topology.
- In abelian groups, the product is the direct product $\prod_{a \in A} X_a$, and the coproduct is the direct sum $\bigoplus_{a \in A} X_a$, which coincides with the product when A is finite.

(2) When $A = \emptyset$, the product (resp. coproduct) admits a unique morphism from each object to it (resp. from it to each object), and hence the terminal (resp. initial) object of the category.

- In sets, the initial object is the empty set and a terminal object can be any singleton set.
- In pointed sets, both the initial a terminal object can be any singleton pointed set.
- In spaces, the initial object is the empty space and a terminal object can be a one-point space.
- In abelian groups, both the initial and terminal object are the trivial group.

Consider the category with a single object with a nontrivial endomorphism on it. In this case, the only object has at least two different morphisms to itself, so it's neither initial nor terminal.

(3) As Y' is a product, such f exists and is unique by definition. On the other hand, as Y is a product as well, there uniquely exists another map $f' : Y' \to Y$ such that $\mathrm{pr}_a \circ f' = \mathrm{pr}'_a$. Combining the two equations on f, f' and $\mathrm{pr}_a, \mathrm{pr}'_a$, we obtain $f \circ f' = \mathrm{id}_{Y'}$ and $f' \circ f = \mathrm{id}_Y$. It follows that f is an isomorphism.

(4) If the product (resp. coproduct) exists in this situation, it must be $\inf A$ (resp. $\sup A$). So the product (resp. coproduct) exists if and only if $\inf A$ (resp. $\sup A$) exists. $\square$

Problem 1.4. Let X, Y be spaces and let $\Pi_1(X)$ and $\Pi_1(Y)$ be their fundamental groupoids. Is it true that any functor $F\colon \Pi_1(X) \to \Pi_1(Y)$ is induced by a continuous map $f\colon X \to Y$?

Solution. It is not true, for which the main idea is as follows. Even though $\Pi_1(X)$ and $\Pi_1(Y)$ have very similar structure so that there is a functor F between them, the topologies of X and Y could be very different.

There can be many counterexamples. For one of them, take $X = Y = \{0, 1\}$ with the same topology $\mathcal{T} \coloneqq \{\emptyset, \{1\}, X\}$, so that $\{0\}$ is not an open subset in X or Y. Consider the functor $F\colon \Pi_1(X) \to \Pi_1(Y)$ with $F(0) = 1$ and $F(1) = 0$, which sends any path to its inverse. Then F is induced by $f\colon X \to Y$ such that $f(0) = 1$ and $f(1) = 0$. In this case, such f is not continuous as $\{0\}$ is open in Y but not open X. $\square$

Homework 2

Problem 2.1 (The Yoneda lemma)**.** Let $F, G\colon \mathcal{C} \to \mathsf{Set}$ be two contravariant functors from a category $\mathcal{C}$ to the category of sets. Suppose F is representable by an object X in $\mathcal{C}$, i.e., there is a natural isomorphism from F to the contravariant functor $C(-,X)\colon \mathcal{C} \to \mathsf{Set}$.

Let $\mathbf{Nat}(F,G)$ be the set of natural transformations from F to G. (You don't need to prove that $\mathbf{Nat}(F,G)$ is a set.) Show that there is a bijection between the sets $\mathbf{Nat}(F,G)$ and $G(X)$.

Solution. By representability of F, we have a bijection between objects in Set:

$$\mathbf{Nat}(F,G) \cong \mathbf{Nat}(C(-,X),G).$$

To prove the Yoneda lemma, we only need to construct a bijection between $\mathbf{Nat}(C(-,X),G)$ and $G(X)$.

First, note that given $\Phi \in \mathbf{Nat}(C(-,X),G)$, the image of $\mathrm{id}_X \in C(X,X)$ gives a unique element $\Phi(\mathrm{id}_X) \in G(X)$. So it remains to deal with the converse direction. Conversely, we show the image of id_X in $G(X)$ uniquely determines an element of $\mathbf{Nat}(C(-,X),G)$, for which we suppose $g_X \in G(X)$ is a provided image of id_X. Indeed, given any $Y \in \mathcal{C}$ together with $(Y \to X) \in C(Y,X)$, we have an induced functor $G(X) \to G(Y)$ as G is contravariant; the image of $g_X \in G(X)$ along this inside $G(Y)$ gives the image of $(Y \to X) \in C(Y,X)$ in $G(Y)$. As a result, the construction gives $C(Y,X) \to G(Y)$ for each $Y \in \mathcal{C}$, depending only on the choice of g_X. This proves the desired bijection between $\mathbf{Nat}(C(-,X),G)$ and $G(X)$. $\square$

Problem 2.2. Consider the isomorphisms

$$H_{n+1}(D^{n+1},S^n) \xrightarrow{\partial} H_n(S^n,*) \xrightarrow{q_*^{-1}} H_n(D^n,S^{n-1}),$$

where ∂ is the boundary map in the long exact sequence for $(D^{n+1},S^n,*)$, and q_* is induced by the quotient map $q\colon D^n \to S^n$. Let $[\iota_n] \in H_n(D^n,S^{n-1})$ be the homology class of the standard homeomorphism

$$\iota_n\colon (\Delta^n,\partial\Delta^n) \xrightarrow{\sim} (D^n,S^{n-1}).$$

Show that

$$(q_*^{-1} \circ \partial)[\iota_{n+1}] = \pm[\iota_n].$$

Solution. By construction, the composite $q \circ \iota_n\colon (\Delta^n,\partial\Delta^n) \to (D^n,S^{n-1}) \to (S^n,*)$ defines the homology class $q_*[\iota_n]$. On the other hand, $\partial[\iota_{n+1}]$ is defined by $(\partial\Delta^{n+1},*) \to (S^n,*)$.

To compare these two classes, we need to rewrite S^n in terms of $\Delta^n/\partial\Delta^n$ and $\partial\Delta^{n+1}$ respectively. For this purpose, choose an embedding

$$d_j\colon \Delta^n \hookrightarrow \partial\Delta^{n+1},$$

whose image is the j-th face of Δ^{n+1} for some $0 \leqslant j \leqslant n+1$. Let $d^j\Delta^{n+1}$ be the j-th coface of Δ^{n+1}, i.e., the union of all but the j-th faces of $\partial\Delta^{n+1}$. Then, up to sign $(-1)^j$, each homology class in $H_n(\Delta^n/\partial\Delta^n,*)$ can be viewed as a homology class in $H_n(\partial\Delta^{n+1}/d^j\Delta^{n+1},*)$, and vice versa. Therefore, up to $(-1)^j$, the following two classes are identified:

$$\partial[\iota_{n+1}] = [(\partial\Delta^{n+1},*) \to (\partial\Delta^{n+1},*)] \in H_n(\partial\Delta^{n+1},*),$$
$$q_*[\iota_n] = [(\Delta^n,\partial\Delta^n) \to (\Delta^n/\partial\Delta^n,*)] \in H_n(\Delta^n/\partial\Delta^n,*).$$

Applying the isomorphism q_*^{-1}, it follows that $(q_*^{-1} \circ \partial)[\iota_{n+1}] = \pm[\iota_n]$. $\square$

Problem 2.3. Solve the following problems.

(1) Let $X = \mathbb{C}^n \setminus \{(0,\cdots,0)\}$. Consider the map

$$f\colon X \longrightarrow X, \quad f(z_1,\cdots,z_n) = (z_1^2,\cdots,z_n^2).$$

Compute the induced map $f_*\colon H_*(X) \to H_*(X)$.

(2) Let $R\colon S^n \to S^n$ be the map that sends a point to its antipodal point. Show that R is homotopic to the identity map if and only if n is odd.

Solution. (1) By construction, X is homotopic to S^{2n-1} by deformation retract, so $H_i(X) \cong \mathbb{Z}$ for $i = 0$ or $2n-1$, and $H_i(X) = 0$ else. It is clear that f_* maps a generator of $H_0(S^{2n-1})$ to another generator with orientation preserved. This forces

$$f_* = \mathrm{id}\colon H_0(X) \longrightarrow H_0(X).$$

Now it suffices to describe $H_{2n-1}(S^{2n-1}) \to H_{2n-1}(S^{2n-1})$ in terms of $\mathbb{Z} \to \mathbb{Z}$. When $n = 1$, the isomorphism $H_1(S^1) \cong \mathbb{Z}$ is given by winding number (see Problem 1.2). In this case, $f\colon z \mapsto z^2$ sends $e^{\pi i\theta}$ to $e^{2\pi i\theta}$, and hence is of degree 2, i.e. f doubles the winding number. So the induced map $f_*\colon H_1(X) \to H_1(X)$ can be described as $\mathbb{Z} \to \mathbb{Z},\ w \mapsto 2w$. Similarly, for general n there is a decomposition $f = f_1 \circ \cdots \circ f_n$, where f_i maps z_i to z_i^2 and preserves z_j for $j \neq i$. It is clear that each f_i is of degree 2, so f is of degree 2^n. The induced homological map $H_{2n-1}(X) \to H_{2n-1}(X)$ is given by $\mathbb{Z} \to \mathbb{Z},\ w \mapsto 2^n w$. To conclude,

$$\begin{array}{ccc} H_{2n-1}(X) \cong \mathbb{Z} & \longrightarrow & \mathbb{Z} \cong H_{2n-1}(X) \\ [\gamma] & \longmapsto & 2^n[\gamma] \end{array}$$

gives the description of f_* for each n.

(2) We describe R over $\mathbb{R}$ through the coordinate system $S^n \hookrightarrow \mathbb{R}^{n+1}$. In this setting, R is given by $(x_1, \ldots, x_{n+1}) \mapsto (-x_1, \ldots, -x_{n+1})$. Then, following the argument in (1), it has degree $(-1)^{n+1}$ as it is the composition of $n+1$ reflections, and the induced map is

$$H_n(S^n) \longrightarrow H_n(S^n), \quad [\gamma] \longmapsto \deg R \cdot [\gamma] = (-1)^{n+1}[\gamma].$$

Thus, if n is even, R cannot be homotopic to the identity map because it induces nontrivial R_*.

In the following, assume $n = 2k+1$ is odd. As in (1), $S^n = S^{2k+1}$ is homotopic to $X = \mathbb{C}^k \setminus \{0\}$ (which does not hold when n is even), so we may identify S^n with X in the following. Now R can be described as the complex map $z = (z_1, \ldots, z_k) \mapsto -z = (-z_1, \ldots, -z_k)$ on $\mathbb{C}^k$, and we construct the homotopy as

$$H\colon X \times I \longrightarrow X, \quad (z, t) \longmapsto e^{\pi i t} z.$$

This H restricts to R when $t = 1$, and restricts to id_X when $t = 0$. So R is homotopic to the identity map on S^n via a complex homotopy whenever n is odd. □

Problem 2.4 (MIT 18.905, Problem Set I, Problem 5)**.**

(1) Let A be a chain complex (of abelian groups). It is *acyclic* if $H(A) = 0$, and *contractible* if it is chain-homotopy-equivalent to the trivial chain complex. Prove that a chain complex is contractible if and only if it is acyclic and for every n the inclusion $Z_nA \to A_n$ is a split monomorphism of abelian groups.

(2) Give an example of an acyclic chain complex that is not contractible.

Solution. (1) Write $d_n\colon A_n \to A_{n-1}$ for differential maps of A.

Assume A is contractible. Then for each n there exists $h_n\colon A_n \to A_{n+1}$ such that the identity chain map at degree n is of the form $\mathrm{id}_{A_n} = h_{n-1}d_n + d_{n+1}h_n$, as given in the following diagram.

$$\begin{array}{ccccccccc} \cdots & \longrightarrow & A_{n+1} & \xrightarrow{d_{n+1}} & A_n & \xrightarrow{d_n} & A_{n-1} & \longrightarrow & \cdots \\ & & \downarrow{\scriptstyle \mathrm{id}} & \swarrow{\scriptstyle h_n} & \downarrow{\scriptstyle \mathrm{id}} & \swarrow{\scriptstyle h_{n-1}} & \downarrow{\scriptstyle \mathrm{id}} & & \\ \cdots & \longrightarrow & A_{n+1} & \xrightarrow{d_{n+1}} & A_n & \xrightarrow{d_n} & A_{n-1} & \longrightarrow & \cdots \end{array}$$

Consider any n-cycle $z_n \in Z_nA$, which by definition is annihilated by d_n. Then $z_n = \mathrm{id}_{A_n}(z_n) = d_{n+1}h_n(z_n) \in \operatorname{im} d_{n+1} = B_nA$. It follows that $Z_nA = B_nA$, and hence $H_n(A) = 0$ for all n, or namely A is acyclic. On the other hand, the image of $d_{n+1}h_n\colon A_n \to A_n$ in contained in Z_nA because $d_nd_{n+1} = 0$. So $d_{n+1}h_n\colon A_n \to Z_nA$ defines a split of $Z_nA \to A_n$, and it clearly becomes id_{Z_nA} after composition with $Z_nA \to A_n$. (See Problem 2.6(1) for the meaning of "split".)

Conversely, assume A is acyclic and every $Z_nA \to A_n$ splits. Then we have a direct decomposition $A_n = Z_nA \oplus A_n'$ for some A_n'; in this case, d_n annihilates Z_nA and maps A_n' to $B_{n-1}A$. But $Z_{n-1}A =$

$B_{n-1}A$ by acyclicity, so we attain an isomorphism $d'_n \coloneqq d_n|_{A'_n} : A'_n \xrightarrow{\sim} Z_{n-1}A$. This gives rise to the construction

$$h_{n-1} \colon A_{n-1} = Z_{n-1}A \oplus A'_{n-1} \xrightarrow{d'^{-1}_n \oplus 0} A_n.$$

It then follows that $h_{n-1}d_n + d_{n+1}h_n = (d'^{-1}_n d'_n)|_{A'_n} + (d_{n+1}d'^{-1}_{n+1})|_{Z_nA} = \mathrm{id}_{A'_n} + \mathrm{id}_{Z_nA} = \mathrm{id}_{A_n}$, showing that A is contractible.

(2) Consider the complex A defined by the short exact sequence $0 \to A_3 \to A_2 \to A_1 \to 0$, where $A_3 = A_2 = \mathbb{Z}$ and $A_1 = \mathbb{Z}/m\mathbb{Z}$ for some $m \in \mathbb{Z}_{\geqslant 2}$. The nonzero differential maps are given by $d_3 \colon x \mapsto mx$ and $d_2 \colon x \mapsto x \bmod m$. Then the complex is identified with

$$A = [0 \longrightarrow \mathbb{Z} \xrightarrow{\times m} \mathbb{Z} \longrightarrow \mathbb{Z}/m\mathbb{Z} \longrightarrow 0].$$

This is clearly acyclic because it is a short exact sequence. However, we have $Z_2A = B_2A = m\mathbb{Z}$, while the quotient $\mathbb{Z}/m\mathbb{Z}$ cannot be viewed as a subgroup of $\mathbb{Z}$; thus, $A_2 = \mathbb{Z}$ cannot split into a direct sum of $Z_2A = m\mathbb{Z}$ and another abelian group. By (1), it follows that A is not contractible. □

Problem 2.5 (MIT 18.905, Problem Set I, Problem 6)**.** Propose a construction of the product and the coproduct of two spaces in the homotopy category, and check that your proposal serves the purpose.

Solution. Recall that the homotopy category hTop is the category of topological spaces with homotopy classes of maps as morphisms. Given a family of objects $\{X_i\}_{i\in I}$, we propose the following constructions:

- The product is $\prod_{i\in I} X_i$, the product of usual topological spaces;
- The coproduct is $\coprod_{i\in I} X_i$, the same as the disjoint union $\sqcup_{i\in I} X_i$ of topological spaces.

To check these, one needs to follow definitions of product and coproduct in Problem 1.3 with some unique factorizations of morphisms considered. For the product, given any $Y \in \mathsf{hTop}$ together with $f_i \colon Y \to X_i$, denote $[f_i]$ the homotopy class of f_i inside hTop. Then the factorization relation $f_i = \mathrm{pr}_i \circ f$ (where $\mathrm{pr}_i \colon \prod_{i\in I} X_i \to X_i$) for the unique map $f \colon Y \to \prod_{i\in I} X_i$ in Top gives $[f_i] = [\mathrm{pr}_i \circ f] = [\mathrm{pr}_i] \circ [f]$ in hTop, with the uniqueness of $[f]$ as well. The verification of coproduct follows from a similar argument. □

Problem 2.6 (MIT 18.905, Problem Set I, Problem 8)**.**

(1) Let $0 \to A \xrightarrow{i} B \xrightarrow{p} C \to 0$ be a short exact sequence. Show that the following three sets are in bijection with one another.
   (a) The set of homomorphisms $\sigma \colon C \to B$ such that $p\sigma = 1_C$.
   (b) The set of homomorphisms $\pi \colon B \to A$ such that $\pi i = 1_A$.
   (c) The set of homomorphisms $\alpha \colon A \oplus C \to B$ such that $\alpha(a, 0) = i(a)$ for all $a \in A$ and $p\alpha(a, c) = c$ for all $(a, c) \in A \oplus C$.

   Moreover, show that any homomorphism as in (c) is an isomorphism. Any one of these structures is a *splitting* of the short exact sequence, and the sequence is then said to be *split*.

(2) Suppose that

$$\begin{array}{ccccccccc} \cdots \longrightarrow & A_n & \longrightarrow & B_n & \longrightarrow & C_n & \longrightarrow & A_{n-1} & \longrightarrow \cdots \\ & \big\downarrow & & \big\downarrow & & \big\downarrow{\scriptstyle\cong} & & \big\downarrow & \\ \cdots \longrightarrow & A'_n & \longrightarrow & B'_n & \longrightarrow & C'_n & \longrightarrow & A'_{n-1} & \longrightarrow \cdots \end{array}$$

   is a "ladder": a map of long exact sequences. So both rows are exact and each square commutes. Suppose also that every third vertical map is an isomorphism, as indicated. Prove that these data naturally determine a long exact sequence

$$\cdots \longrightarrow A_n \longrightarrow A'_n \oplus B_n \longrightarrow B'_n \longrightarrow A_{n-1} \longrightarrow \cdots$$

Solution. (1) By definition of short exact sequence, we obtain $pi = 0$ as well as $A \cong \ker p$ and $C \cong \operatorname{coker} i$. It is enough to show that each of (a) and (b) is in bijection with (c).

For the bijection between (a) and (c), given such $\sigma \colon C \to B$ in (a), rewrite each $b \in B$ as

$$b = (b - \sigma p(b)) + \sigma p(b).$$

It is clear that $\sigma p(b) \in \operatorname{im}\sigma$. On the other hand, since $p\sigma = 1_C$, we have $p(b - \sigma p(b)) = p(b) - (p\sigma)(p(b)) = p(b) - p(b) = 0 \in C$, and hence $b - \sigma p(b) \in \ker p$. Note that $p\sigma = 1_C$ also implies $\ker p \cap \operatorname{im}\sigma = 0$. (Indeed, if this was nonzero, there would exist $x \in B$ such that $x = \sigma(y)$ for some nonzero $y \in C$, but the condition forces a contradiction as $0 = p(x) = p(\sigma(y)) = y$.) So the assignment $b \mapsto (b - \sigma p(b), \sigma p(b))$ defines the direct decomposition

$$B \cong \ker p \oplus \operatorname{im}\sigma.$$

In this direct decomposition, we know $\ker p \cong A$ and aim to show $\operatorname{im}\sigma \cong C$, for which we only need to show σ is injective; but $\ker\sigma \subset \ker(p\sigma) = \ker 1_C = 0$, so we have proved that $B \cong A \oplus C$. Now we construct

$$\alpha\colon A \oplus C \xrightarrow{\sim} B, \quad (a, c) \longmapsto i(a) + \sigma(c).$$

Then $\alpha(a, 0) = i(a)$ and $p(\alpha(a, c)) = pi(a) + p\sigma(c) = c$. From the argument above, if σ in (a) exists then the corresponding α in (c) must be an isomorphism. This completes the construction from (a) to (c). Conversely, given α in (c), taking σ as the composite of natural inclusion $C \hookrightarrow A \oplus C$ with α suffices.

As for the bijection between (b) and (c), apply the same argument to show $B \cong A \oplus C$ but with rewriting each $b \in B$ as $b = (b - i\pi(b)) + i\pi(b) \in \ker\pi \oplus \operatorname{im} i$.

(2) We label the maps as follows.

$$\begin{array}{ccccccccc}
\cdots \longrightarrow & A_n & \xrightarrow{d_n} & B_n & \xrightarrow{t_n} & C_n & \xrightarrow{\delta_n} & A_{n-1} & \longrightarrow \cdots \\
 & \downarrow{\scriptstyle \alpha_n} & & \downarrow{\scriptstyle \beta_n} & & \downarrow{\scriptstyle \gamma_n}\,\cong & & \downarrow{\scriptstyle \alpha_{n-1}} & \\
\cdots \longrightarrow & A'_n & \xrightarrow{d'_n} & B'_n & \xrightarrow{t'_n} & C'_n & \xrightarrow{\delta'_n} & A'_{n-1} & \longrightarrow \cdots
\end{array}$$

In the desired long exact sequence, for each n, we construct

$$q_n\colon A_n \longrightarrow A'_n \oplus B_n, \quad a \longmapsto (\alpha_n(a), d_n(a))$$
$$p_n\colon A'_n \oplus B_n \longrightarrow B'_n, \quad (a', b) \longmapsto d'_n(a) - \beta_n(b).$$

Also, the connecting map $B'_n \to A_{n-1}$ is given by

$$r_n := \delta_n \circ \gamma_n^{-1} \circ t'_n\colon B'_n \longrightarrow A_{n-1}.$$

We need to verify that the composite of any two consecutive maps in the long exact sequence equals 0, i.e. we need

$$r_n \circ p_n = 0, \qquad q_{n-1} \circ r_n = 0.$$

By construction we have $p_n \circ q_n = 0$, because of $\beta_n \circ d_n = d'_n \circ \alpha_n$ by commutativity. Next, note that $t'_n \circ d'_n = 0$ and $\delta_n \circ t_n = 0$, where the latter relation then implies $\delta_n \circ \gamma_n^{-1} \circ t'_n \circ \beta_n = 0$; so we deduce that $r_n \circ p_n = 0$. Further, since $d_{n-1} \circ \delta_n = 0$, in order to check $q_{n-1} \circ r_n = 0$, it suffices to check $\alpha_{n-1} \circ r_n = 0$; but by commutativity we have $\alpha_{n-1} \circ \delta_n = \delta'_n \circ \gamma_n$, which gives $\alpha_{n-1} \circ r_n = \delta'_n \circ \gamma_n \circ \gamma_n^{-1} \circ t'_n = \delta'_n \circ t'_n = 0$.

Now it remains to check the exactness at each term. For this, by construction $\ker p_n \subset \operatorname{im} q_n$, $\ker r_n \subset \operatorname{im} p_n$, and $\ker q_{n-1} \subset \operatorname{im} r_n$; these inclusions automatically upgrade to equalities because $p_n \circ q_n = r_n \circ p_n = q_{n-1} \circ r_n = 0$. This finishes the proof that the desired sequence is exact. □

Problem 2.7 (MIT 18.905, Problem Set I, Problem 10)**.**

> This exercise generalizes our computation of the homology of spheres, and introduces several important constructions.

The *cone* on a space X is the quotient space $CX = (X \times I)/(X \times \{0\})$, where I is the unit interval $[0, 1]$. The cone is a pointed space, with basepoint $*$ given by the "cone point", i.e., the image of $X \times \{0\}$. (By convention, the cone on the empty space $\varnothing$ is a single point, the cone point.) Regard X as the subspace of CX of all points of the form $(x, 1)$.

Define the *suspension* of a space X to be $SX = CX/X$. Make SX a pointed space by declaring the image of $X \subset CX$ to be the basepoint in SX. (By convention, the quotient $W/\varnothing$ is the disjoint union of W with a single point, which is declared to be the basepoint. So $S\varnothing = */\varnothing$ is the discrete two-point space, with the new point as basepoint.)

The quotient map induces a map of pairs $f\colon (CX, X) \to (SX, *)$.

(1) Show that CX is contractible.
(2) Show that there is a natural isomorphism $\tilde{H}_{n-1}(X) \to H_n(SX, *)$, for any n.

Solution. (1) Denote $[x, t]$ the representative of point $(x, t) \in X \times I$ in the quotient CX. Consider the homotopy

$$H\colon CX \times I \longrightarrow CX, \quad ([x, t], s) \longmapsto [x, (1 - s)t].$$

It is direct to check that $(1 - s)t \in I$ for all $s, t \in I$. From the construction of H, notice that $H([x, t], 0) = [x, t]$ and $H([x, t], 1) = [x, 0] \in X \times \{0\}$; so $H|_{CX\times\{0\}} = \mathrm{id}_{CX}$ and $H([x, t], 1)$ is the basepoint of CX. It follows that CX is contractible.

(2) Note that we have a short exact sequence of chain complexes:

$$0 \longrightarrow S_*(X) \longrightarrow S_*(CX) \longrightarrow S_*(CX, X) \longrightarrow 0.$$

This further induces a homological long exact sequence

$$\cdots \longrightarrow H_{n+1}(CX, X) \longrightarrow \tilde{H}_n(X) \longrightarrow \tilde{H}_n(CX) \longrightarrow H_n(CX, X) \longrightarrow \cdots$$

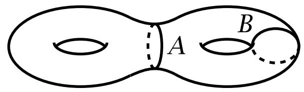
in which $\tilde{H}_n(CX) = 0$ for each n as CX is contractible by (1). Thus, for each n, we obtain an isomorphism $H_{n+1}(CX, X) \cong \tilde{H}_n(X)$. Replacing n with $n - 1$, it suffices to figure out the natural isomorphism $H_n(CX, X) \cong H_n(SX, *)$. But this is clearly given by $H_n(CX, X) \cong \tilde{H}_n(CX/X) = H_n(SX, *)$. $\square$

Homework 3

Problem 3.1 (Hatcher, §2.1, Problem 17)**.**

(1) Compute the homology groups $H_n(X, A)$ when X is S^2 or $S^1 \times S^1$ and A is a finite set of points in X.

(2) Compute the groups $H_n(X, A)$ and $H_n(X, B)$ for X a closed orientable surface of genus 2 with A and B the circles shown.



Solution. (1) For $X = S^2$ or $S^1 \times S^1$ of dimension 2, we always have the long exact sequence

$$\begin{array}{l} 0 \longrightarrow H_2(A) \longrightarrow H_2(X) \longrightarrow H_2(X,A) \\ \quad \hookrightarrow H_1(A) \longrightarrow H_1(X) \longrightarrow H_1(X,A) \\ \quad \hookrightarrow H_0(A) \longrightarrow H_0(X) \longrightarrow H_0(X,A) \longrightarrow 0. \end{array}$$

For simplicity, assume $A \neq \emptyset$ so that $|A| \geqslant 1$. Note that $H_0(A) \cong \mathbb{Z}^{\oplus|A|}$ and $H_1(A) = H_2(A) = 0$. As for $H_n(X)$, we know the following.

(a) When $X = S^2$, we have $H_0(X) \cong H_2(X) \cong \mathbb{Z}$ and $H_1(X) = 0$.

(b) When $X = S^1 \times S^1$, we have $H_0(X) \cong H_2(X) \cong \mathbb{Z}$ and $H_1(X) \cong \mathbb{Z} \oplus \mathbb{Z}$.

Thus, in case (a) and (b), the long exact sequence above respectively becomes

$$\begin{array}{l} 0 \longrightarrow 0 \longrightarrow \mathbb{Z} \longrightarrow H_2(X,A) \\ \quad \hookrightarrow 0 \longrightarrow 0 \longrightarrow H_1(X,A) \\ \quad \hookrightarrow \mathbb{Z}^{\oplus|A|} \longrightarrow \mathbb{Z} \longrightarrow H_0(X,A) \longrightarrow 0, \end{array} \qquad \begin{array}{l} 0 \longrightarrow 0 \longrightarrow \mathbb{Z} \longrightarrow H_2(X,A) \\ \quad \hookrightarrow 0 \longrightarrow \mathbb{Z}\oplus\mathbb{Z} \longrightarrow H_1(X,A) \\ \quad \hookrightarrow \mathbb{Z}^{\oplus|A|} \longrightarrow \mathbb{Z} \longrightarrow H_0(X,A) \longrightarrow 0. \end{array}$$

For either case, X is path-connected so that $H_0(X, A) = 0$. Also, in either case the first row of the sequence deduces that $H_2(X, A) \cong \mathbb{Z}$. So it remains to compute $H_1(X, A)$. For the former case $X = S^2$, we obtain a short exact sequence $0 \to H_1(S^2, A) \to \mathbb{Z}^{\oplus|A|} \to \mathbb{Z} \to 0$, so there must be $H_1(S^2, A) \cong \mathbb{Z}^{\oplus|A|-1}$. As for the latter case $X = S^1 \times S^1$, provided with the exact sequence $0 \to \mathbb{Z} \oplus \mathbb{Z} \to H_1(X, A) \to \mathbb{Z}^{\oplus|A|} \to \mathbb{Z} \to 0$, the image of $H_1(X, A)$ in $\mathbb{Z}^{\oplus|A|}$ should be equal to $\ker(\mathbb{Z}^{\oplus|A|} \to \mathbb{Z})$, so we deduce $H_1(S^1 \times S^1, A) \cong \mathbb{Z}^{\oplus|A|+1}$. In summary, when $A \neq \emptyset$, we have

$$H_n(S^2, A) \cong \begin{cases} \mathbb{Z}, & n = 2, \\ \mathbb{Z}^{\oplus|A|-1}, & n = 1, \\ 0, & n = 0; \end{cases} \qquad H_n(S^1 \times S^1, A) \cong \begin{cases} \mathbb{Z}, & n = 2, \\ \mathbb{Z}^{\oplus|A|+1}, & n = 1, \\ 0, & n = 0. \end{cases}$$

This finishes the desired computation.

(2) Both (X, A) and (X, B) are good pairs, so $H_n(X, A) \cong \tilde{H}_n(X/A)$ and $H_n(X, B) \cong \tilde{H}_n(X/B)$ for all n. Notice that X/A is the union of two tori of genus 1 with one point connected, so we have $X/A \simeq (S^1 \times S^1) \vee (S^1 \times S^1)$. Applying the result of part (1) renders that

$$H_n(X, A) \cong \tilde{H}_n(X/A) \cong H_n(S^1 \times S^1, *)^{\oplus 2} \cong \begin{cases} \mathbb{Z}^{\oplus 2}, & n = 2, \\ \mathbb{Z}^{\oplus 4}, & n = 1, \\ 0, & n = 0. \end{cases}$$

On the other hand, X/B is homeomorphic to a torus with two points identified, that is, $X/B \simeq (S^1 \times S^1)/\{x_1, x_2\}$. It follows that

$$H_n(X,B) \cong \tilde{H}_n(X/B) \cong H_n(S^1 \times S^1, \{x_1,x_2\}) \cong \begin{cases} \mathbb{Z}, & n=2, \\ \mathbb{Z}^{\oplus 3}, & n=1, \\ 0, & n=0. \end{cases}$$

For $n \geqslant 3$, the homology vanishes as X is of dimension 2. □

Problem 3.2 (Hatcher, §2.1, Problem 27)**.** Let $f\colon (X,A) \to (Y,B)$ be a map such that both $f\colon X \to Y$ and the restriction $f\colon A \to B$ are homotopy equivalences.

(1) Show that $f_*\colon H_n(X,A) \to H_n(Y,B)$ is an isomorphism for all n.
(2) For the case of the inclusion $f\colon (D^n, S^{n-1}) \hookrightarrow (D^n, D^n - \{0\})$, show that f is not a homotopy equivalence of pairs — there is no $g\colon (D^n, D^n-\{0\}) \to (D^n, S^{n-1})$ such that fg and gf are homotopic to the identity through maps of pairs.

Solution. (1) We always have short exact sequence of chain complexes $0 \to S_*(A) \to S_*(X) \to S_*(X,A) \to 0$ and similarly for (Y,B). Taking homology on these, the map $f\colon (X,A) \to (Y,B)$ induces that

$$\begin{array}{ccccccccc} \cdots \longrightarrow & H_n(A) & \longrightarrow & H_n(X) & \longrightarrow & H_n(X,A) & \longrightarrow & H_{n-1}(A) & \longrightarrow \cdots \\ & \downarrow{\cong} & & \downarrow{\cong} & & \downarrow{f_*} & & \downarrow{\cong} & \\ \cdots \longrightarrow & H_n(B) & \longrightarrow & H_n(Y) & \longrightarrow & H_n(Y,B) & \longrightarrow & H_{n-1}(B) & \longrightarrow \cdots \end{array}$$

where $H_n(A) \cong H_n(B)$ and $H_n(X) \cong H_n(Y)$ for all n by our homotopy assumption. So $f_*\colon H_n(X,A) \to H_n(Y,B)$ is forced to be an isomorphism for all n.

(2) Assume $f\colon (D^n, S^{n-1}) \hookrightarrow (D^n, D^n - \{0\})$ is a homotopy equivalence for the sake of contradiction. Then by replacing S^{n-1} and $D^n - \{0\}$ with their closures S^{n-1} and D^n, respectively, we get another homotopy

$$(D^n, S^{n-1}) \xrightarrow{\sim} (D^n, D^n).$$

Then by (1), there must be an isomorphism $H_i(D^n, S^{n-1}) \cong H_i(D^n, D^n)$ for each i. But for $i = n$, the former is $H_n(D^n/S^{n-1}, *) \cong H_n(S^n) \cong \mathbb{Z}$ while the latter is trivial, leading to a contradiction. So the given map f cannot be a homotopy equivalence. □

Problem 3.3 (Hatcher, §2.1, Problem 29)**.** Show that $S^1 \times S^1$ and $S^1 \vee S^1 \vee S^2$ have isomorphic homology groups in all dimensions, but their universal covering spaces do not.

Solution. By property of wedge product, $\tilde{H}_n(S^1 \vee S^1 \vee S^2) \cong \tilde{H}_n(S^1) \oplus \tilde{H}_n(S^1) \oplus \tilde{H}_n(S^2)$. As $H_n(S^m) \cong \mathbb{Z}$ for $n = 0, m$ and it vanishes for $n \neq 0, m$, we obtain

$$H_n(S^1 \vee S^1 \vee S^2) \cong \begin{cases} \mathbb{Z}, & n = 0,2, \\ \mathbb{Z} \oplus \mathbb{Z}, & n=1, \\ 0, & \text{else}. \end{cases}$$

This homology group is clearly the same as that of torus, which is identified with $H_n(S^1 \times S^1)$. So the two given spaces have isomorphic homology groups.

We then consider the universal covering spaces as follows.

(a) For $S^1 \times S^1 \simeq \mathbb{R}/\mathbb{Z} \times \mathbb{R}/\mathbb{Z} \simeq \mathbb{R}^2/\mathbb{Z}^2$, the universal covering space is clearly $\mathbb{R}^2$.
(b) For $S^1 \vee S^1 \vee S^2$, fix the basepoint as the connecting point. Notice that $S^1 \vee S^1 \simeq \mathbb{R}/\mathbb{Z} \vee \mathbb{R}/\mathbb{Z}$ and S^2 itself is simplify connected. So the universal covering is $(\mathbb{R} \cup_{\{0\}} \mathbb{R}) \cup_{\mathbb{Z}\times\mathbb{Z}} S^2$.

In (a), the universal covering $\mathbb{R}^2$ is contractible and hence $H_2(\mathbb{R}^2) = 0$, whereas $H_2(S^2) \cong \mathbb{Z}$ gives a direct summand of the universal covering in (b) and hence $H_2((\mathbb{R} \cup_{\{0\}} \mathbb{R}) \cup_{\mathbb{Z}\times\mathbb{Z}} S^2) \neq 0$. To conclude, the two given spaces have different universal coverings because their H_2's are different. □

Problem 3.4 (Hatcher, §2.2, Problem 2)**.** Given a map $f\colon S^{2n} \to S^{2n}$, show that there is some point $x \in S^{2n}$ with either $f(x) = x$ or $f(x) = -x$. Deduce that every map $\mathbb{RP}^{2n} \to \mathbb{RP}^{2n}$ has a fixed point. Construct maps $\mathbb{RP}^{2n-1} \to \mathbb{RP}^{2n-1}$ without fixed points from linear transformations $\mathbb{R}^{2n} \to \mathbb{R}^{2n}$ without eigenvectors.

Solution. Assume f has no fixed points. Under this assumption, recall the property that $\deg f = (-1)^{2n+1} = -1$. On the other hand, the antipodal map $-\mathbb{1}\colon S^{2n} \to S^{2n},\ x \mapsto -x$ has degree $(-1)^{2n+1} = -1$ as well, so we have $\deg(-f) = \deg(-\mathbb{1}) \cdot \deg f = 1$. It follows that $-f$ has a fixed point $x \in S^{2n}$ such that $(-f)(x) = -f(x) = x$, or equivalently $f(x) = -x$.

Note that $\mathbb{RP}^{2n}$ is a quotient of S^{2n} by gluing the antipodal points. So S^{2n} is a covering space of $\mathbb{RP}^{2n}$, encoded by the natural map $p\colon S^{2n} \to \mathbb{RP}^{2n}$. Let $\varphi\colon \mathbb{RP}^{2n} \to \mathbb{RP}^{2n}$ be any map. Then we obtain $\varphi \circ p\colon S^{2n} \to \mathbb{RP}^{2n}$, which lifts to the covering of its target space, i.e. there exists $f\colon S^{2n} \to S^{2n}$ such that $\varphi \circ p = p \circ f$; in other words, the following diagram commutes:

$$\begin{array}{ccc} S^{2n} & \xrightarrow{\ f\ } & S^{2n} \\ \big\downarrow{\scriptstyle p} & & \big\downarrow{\scriptstyle p} \\ \mathbb{RP}^{2n} & \xrightarrow{\ \varphi\ } & \mathbb{RP}^{2n}. \end{array}$$

Using the previous assertion, there is $x \in S^{2n}$ with either $f(x) = x$ or $f(x) = -x$. By construction, along p both x and $-x$ have the same image in $\mathbb{RP}^{2n}$. It follows that $p(f(x)) = p(x)$, and then by lifting property $p(x) = \varphi(p(x))$. This means $p(x)$ is a fixed point of φ.

For the last task, begin with the linear transform $\mathbb{C}^n \to \mathbb{C}^n,\ z \mapsto iz$. This defines a rotation $r\colon \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ if we identify $\mathbb{C}^n$ with $\mathbb{R}^{2n}$, and it has no eigenvalue over $\mathbb{R}$ because it maps $\mathbb{R}$ to $i\mathbb{R}$. Then we get an induced map on $\mathbb{RP}^{2n-1} = (\mathbb{R}^{2n} - \{0\})/\sim$ from r on $\mathbb{R}^{2n}$. As r has no real eigenvectors, the induced map $\mathbb{RP}^{2n-1} \to \mathbb{RP}^{2n-1}$ has no fixed points. □

Problem 3.5 (Hatcher, §2.2, Problem 3)**.** Let $f\colon S^n \to S^n$ be a map of degree zero. Show that there exist points $x, y \in S^n$ with $f(x) = x$ and $f(y) = -y$. Use this to show that if F is a continuous vector field defined on the unit ball D^n in $\mathbb{R}^n$ such that $F(x) \neq 0$ for all x, then there exists a point on ∂D^n where F points radially outward and another point on ∂D^n where F points radially inward.

Solution. Recall the property that if $f\colon S^n \to S^n$ has no fixed points then $\deg f = (-1)^{n+1}$. In our case, by assumption $\deg f = 0$, so f has a fixed point $x \in S^n$. Next, notice that $f(y) = -y$ is equivalent to $(-f)(y) = y$, so it suffices to find a fixed point y for $-f$. But antipodal map $S^n \to S^n,\ x \mapsto -x$ has degree $(-1)^{n+1}$, implying that $\deg(-f) = (-1)^{n+1} \deg f = 0$. It follows that the desired y exists as well.

Now we normalize $F\colon D^n \to \mathbb{R}^n$ by considering $F/\|F\|$. The normalized map a priori has target $\mathbb{R}^n$ but clearly factors through S^{n-1} as its image has norm 1. So we obtain $F/\|F\|\colon D^n \to S^{n-1} = \partial D^n$. Consider the homeomorphism $S^n \simeq D^n \cup_{\partial D^n} D^n$ by gluing two D^n's as two hemispheres of S^n. Then $F/\|F\|$ gives rise to a composite

$$f\colon S^n \longrightarrow S^{n-1} = \partial D^n \hookrightarrow S^n,$$

in which the restriction of the first map $S^n \to S^{n-1}$ to each D^n is required to be $F/\|F\|$. This f is not surjective as it factors through S^{n-1}, so $\deg f = 0$. By the previous assertion, there are points $x, y \in S^n$ such that $f(x) = x$ and $f(y) = -y$. Again, as f factors through $S^{n-1} = \partial D^n$, the images $x, -y$ land in ∂D^n, and hence $x, y \in \partial D^n \subset S^n$ as well. To conclude, since $F \neq 0$ pointwise, there are $\lambda, \mu > 0$ such that on ∂D^n the vector field F maps x (resp. y) to λx (resp. $-\mu y$), and hence F points radically outward at x (resp. inward at y). □

Problem 3.6 (Hatcher, §2.2, Problem 7)**.** For an invertible linear transformation $f\colon \mathbb{R}^n \to \mathbb{R}^n$, show that the induced map on $H_n(\mathbb{R}^n, \mathbb{R}^n - \{0\}) \cong \tilde{H}_{n-1}(\mathbb{R}^n - \{0\}) \cong \mathbb{Z}$ is $\mathbb{1}$ or $-\mathbb{1}$ according to whether the determinant of f is positive or negative.

Solution. Let $T_f \in \mathrm{M}_n(\mathbb{R})$ be a matrix representation of f. Since f is invertible, all eigenvalues of T_f are nonzero. By Gaussian elimination, T_f is to be converted into a diagonal matrix D_f with ± 1's on the diagonal, after multiplying a series of elementary matrices on the left of T_f. Notice that all elementary matrices can be continuously deformed into the identity matrix I_n. So there is a path in $\mathrm{M}_n(\mathbb{R})$ from T_f to D_f. In other words, relative to $\mathbb{R}^n - \{0\}$, the given map f is homotopic to $f_{\pm 1}\colon \mathbb{R}^n \to \mathbb{R}^n$ that multiplies each coordinate component by 1 or -1.

Assume the numbers of 1's and -1's on the diagonal of D_f are respectively k and $n-k$. Using the mapping-degree argument (see Problem 2.3), the induced map $f_{\pm 1,*}\colon H_n(\mathbb{R}^n, \mathbb{R}^n - \{0\}) \to H_n(\mathbb{R}^n, \mathbb{R}^n - \{0\})$ in terms of $\mathbb{Z} \to \mathbb{Z}$ has degree $1^k \cdot (-1)^{n-k} = \det D_f = \det T_f \in \{\pm 1\}$. So the original induced map f_* must be $\pm \mathbb{1}$ depending on the sign of $\det T_f$. □

Homework 4

Problem 4.1 (Hatcher, §2.2, Problem 8)**.** A polynomial $f(z)$ with complex coefficients, viewed as a map $\mathbb{C} \to \mathbb{C}$, can always be extended to a continuous map of one-point compactifications $\hat{f}\colon S^2 \to S^2$. Show that the degree of $\hat{f}$ equals the degree of f as a polynomial. Show also that the local degree of $\hat{f}$ at a root of f is the multiplicity of the root.

Solution. Suppose f is a polynomial of degree d and write $f(z) = a_d z^d + \cdots + a_1 z + a_0$. Then there is a homotopy $\mathbb{C} \times I \to \mathbb{C}$, $(z,t) \mapsto t(f(z) - a_d z^d) + a_d z^d$ deforming $f(z)$ to $a_d z^d$. Thus, without loss of generality, we may assume $f(z) = a_d z^d$ for simplicity. This f extends to a map $\hat{f}\colon S^2 \to S^2$ continuously. Then by continuity, for z near ∞ we have $\hat{f}(z) \approx a_d z^d$ with lower order terms omitted. It follows that $\hat{f}$ has local degree d at ∞. On the other hand, the mapping degree of $\hat{f}$ is equal to the local degree at ∞, which is $d = \deg f$. This proves the first assertion.

For the second assertion, let $\alpha \in \mathbb{C}$ be a root of f. Then there exists $c(\alpha) \neq 0$ and $m(\alpha) \in \mathbb{Z}_{\geqslant 1}$ such that $f(z) \approx c(\alpha)(z-\alpha)^{m(\alpha)}$ for z near α. Using the extension argument above, the local degree of $\hat{f}$ at the image of α in S^2 should be $m(\alpha)$, equal to the multiplicity of α. □

Problem 4.2 (Hatcher, §2.2, Problem 10)**.** Let X be the quotient space of S^2 under the identifications $x \sim -x$ for x in the equator S^1. Compute the homology groups $H_i(X)$. Do the same for S^3 with antipodal points of the equatorial $S^2 \subset S^3$ identified.

Solution. By construction, X is homeomorphic to the space by gluing two D^2's along $\mathbb{RP}^1$, and each D^2 wraps around the $\mathbb{RP}^1$ twice; the wrapping number here is computed by $1 + \deg(-\mathbb{1}) = 2$ where $-\mathbb{1}\colon S^1 \to S^1$ is the antipodal map. So X is the same as the space by gluing two copies of $\mathbb{RP}^2$'s along $\mathbb{RP}^1$. Thus, X has the cell structure with a 0-cell c_0, a 1-cell c_1, together with two 2-cells $c_{2,\pm}$; here $c_{2,\pm}$ respectively correspond to the upper and lower hemispheres of S^2. Then the cellular complex has the form

$$0 \longrightarrow \mathbb{Z}\langle c_{2,+}, c_{2,-}\rangle \xrightarrow{\partial_2} \mathbb{Z}\langle c_1\rangle \xrightarrow{\partial_1} \mathbb{Z}\langle c_0\rangle \longrightarrow 0.$$

Note that $\partial_1 = 0$ and $\partial_2(c_{2,\pm}) = (1 + \deg(-\mathbb{1})) \cdot c_1 = 2c_1$. Taking homology on this cellular complex, we obtain

$$H_i(X) \cong \begin{cases} \mathbb{Z}, & i = 2, \\ \mathbb{Z}/2\mathbb{Z}, & i = 1, \\ \mathbb{Z}, & i = 0. \end{cases}$$

This gives the desired homology of X in the S^2-case.

Next, we consider the case of S^3. In this case, X is obtained by gluing two D^3's along $\mathbb{RP}^2$. There is a 0-cell c_0, a 1-cell c_1, a 2-cell c_2, as well as two 3-cells $c_{3,\pm}$. Now the cellular complex is read as

$$0 \longrightarrow \mathbb{Z}\langle c_{3,+}, c_{3,-}\rangle \xrightarrow{\partial_3} \mathbb{Z}\langle c_2\rangle \xrightarrow{\partial_2} \mathbb{Z}\langle c_1\rangle \xrightarrow{\partial_1} \mathbb{Z}\langle c_0\rangle \longrightarrow 0.$$

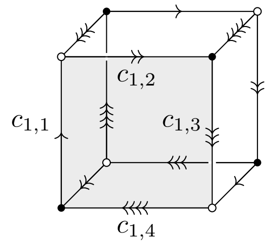
Since X has odd dimension, $\partial_3 = 0$; for the remaining two maps, as before, we obtain $\partial_1 = 0$, and $\partial_2(c_2) = (1 + \deg(-\mathbb{1})) \cdot c_1 = 0$ because the new antipodal map is given by $-\mathbb{1}\colon S^2 \to S^2$. Taking homology on this cellular complex, we obtain

$$H_i(X) \cong \begin{cases} \mathbb{Z} \oplus \mathbb{Z}, & i = 3, \\ 0, & i = 2, \\ \mathbb{Z}/2\mathbb{Z}, & i = 1, \\ \mathbb{Z}, & i = 0. \end{cases}$$

This gives the desired homology of X in the S^3-case. □

Problem 4.3 (Hatcher, §2.2, Problem 11)**.** In [Hat02, §1.2, Exercise 14] we described a 3-dimensional CW complex obtained from the cube I^3 by identifying opposite faces via a one-quarter twist. Compute the homology groups of this complex.

Solution. Recall the construction of [Hat02, §1.2, Exercise 14] as follows. In a cube I^3, identify each square face with the opposite square face via the right-handed screw motion of 90°. For example, if two opposite square faces are labeled on vertices by $ABCD$ and $A'B'C'D'$, then the quotient space identifies A, B, C, D with B', C', D', A', respectively. Also, [Hat02, §1.2, Exercise 14] claims that there are two 0-cells $c_{0,1}, c_{0,2}$, four 1-cells $c_{1,1}, c_{1,2}, c_{1,3}, c_{1,4}$, three 2-cells $c_{2,1}, c_{2,2}, c_{2,3}$, and one 3-cell c_3. Indeed, this cellular structure is induced from that on I^3 by quotient, and the cells can be counted through the following picture.



Then the cellular complex has the form

$$0 \longrightarrow \mathbb{Z}\langle c_3\rangle \xrightarrow{\partial_3} \mathbb{Z}\langle c_{2,j}\rangle_{j=1}^3 \xrightarrow{\partial_2} \mathbb{Z}\langle c_{1,j}\rangle_{j=1}^4 \xrightarrow{\partial_1} \mathbb{Z}\langle c_{0,j}\rangle_{j=1}^2 \longrightarrow 0.$$

For dimension reason, $\partial_3 = 0$, which immediately implies that $H_3(X) \cong \mathbb{Z}$. Notice that ∂_1 sends each $c_{1,j}$ to either $c_{0,1} - c_{0,2}$ or $c_{0,2} - c_{0,1}$, and hence $\operatorname{im}\partial_1 = \mathbb{Z}\cdot(c_{0,1} - c_{0,2}) \cong \mathbb{Z}$. It follows that $H_0(X) = \mathbb{Z}^{\oplus 2}/\operatorname{im}\partial_1 \cong \mathbb{Z}$. As for $H_2(X)$, note that ∂_2 maps each $c_{2,j}$ to the boundary; more explicitly, according to the picture above, we have

$$\begin{aligned}
\partial_2(c_{2,1}) &= c_{1,1} + c_{1,2} + c_{1,3} + c_{1,4},\\
\partial_2(c_{2,2}) &= c_{1,1} + c_{1,2} - c_{1,3} - c_{1,4},\\
\partial_2(c_{2,3}) &= c_{1,1} - c_{1,2} - c_{1,3} + c_{1,4}.
\end{aligned}$$

Thus, ∂_2 can be written in terms of coordinates over $\mathbb{Z}$ as $\partial_2\colon (x, y, z) \mapsto (x+y+z, x+y-z, x-y-z, x-y+z)$; in particular, ∂_2 is injective, and hence $H_2(X) = 0$ follows.

Now it remains to compute $H_1(X)$. By the argument above, $\ker\partial_1$ is generated by elements of the form $c_{1,j} + c_{1,j+1}$ (where we read the subscripts modulo 4); in other words, $\ker\partial_1$ is generated in $\mathbb{Z}^{\oplus 4}$ by $\gamma_{12} = (1,1,0,0)$, $\gamma_{23} = (0,1,1,0)$, $\gamma_{34} = (0,0,1,1)$, $\gamma_{41} = (1,0,0,1)$. On the other hand, $\operatorname{im}\partial_2$ is generated in $\mathbb{Z}^{\oplus 4}$ by elements $(1,1,1,1)$, $(1,1,-1,-1)$, $(1,-1,-1,1)$. So in $H_1(X)$ there must be $\gamma_{12} = \gamma_{34}$ and $\gamma_{23} = \gamma_{41}$, and hence all γ_{ij}'s above have order 2. Since $\gamma_{12} + \gamma_{23} \notin \operatorname{im}\partial_2$, we conclude $H_1(X)$ is a group of order 4 with two elements of order 2, and hence isomorphic to $\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$, where the two generators can be γ_{12} and γ_{23}. To sum up, we obtain

$$H_i(X) \cong \begin{cases} \mathbb{Z}, & i = 3,\\ 0, & i = 2,\\ \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}, & i = 1,\\ \mathbb{Z}, & i = 0. \end{cases}$$

This is the homology of the cellular complex. □

Problem 4.4 (Hatcher, §2.2, Problem 14)**.** A map $f\colon S^n \to S^n$ satisfying $f(x) = f(-x)$ is called an *even map*. Show that an even map $S^n \to S^n$ must have even degree, and that the degree must in fact be zero when n is even. When n is odd, show there exist even maps of any given even degree.

Solution. Suppose $f\colon S^n \to S^n$ is an even map. Then the antipodal points in S^n have the same image along f, and hence f factors through $\mathbb{RP}^n$, i.e. there exists $g\colon \mathbb{RP}^n \to S^n$ such that $f = g \circ p$, where $p\colon S^n \to \mathbb{RP}^n$ is the canonical quotient map identifying x with $-x$. In other words, we obtain a commutative diagram:

$$\begin{array}{ccc} S^n & \xrightarrow{p} & \mathbb{RP}^n \\ & {\scriptstyle f}\searrow & \downarrow{\scriptstyle g} \\ & & S^n. \end{array}$$

By functoriality, this factorization induces the maps between homology groups:

$$f_*\colon H_n(S^n) \xrightarrow{p_*} H_n(\mathbb{RP}^n) \xrightarrow{g_*} H_n(S^n).$$

Recall that $H_n(S^n) \cong \mathbb{Z}$. Now we split the argument on $H_n(\mathbb{RP}^n)$ into the following two cases, so as to prove that f has an even degree.

(i) When n is even, we have $H_n(\mathbb{RP}^n) = 0$. In this case, f_* is the zero map, implying that f must have degree 0.

(ii) When n is odd, we have $H_n(\mathbb{RP}^n) \cong \mathbb{Z}$. This $H_n(\mathbb{RP}^n)$ is generated by the unique n-cell in $\mathbb{RP}^n$. Since p_* maps the n-cell c_n in S^n to $2c_n$ in $\mathbb{RP}^n$, we see p has degree 2, and hence $\deg f = 2 \deg g$ must be even.

Now it remains to show the existence of even maps of any given even degree when n is odd. By (ii) above, it suffices to show for any given integer $k \geqslant 0$ there is $g\colon \mathbb{RP}^n \to S^n$ of degree k. As n is odd, $\mathbb{RP}^n = (\mathbb{R}^{n+1} - \{0\})/\sim$ can be described by complex coordinate $(z_1, \ldots, z_{(n+1)/2})$ in $\mathbb{C}^{(n+1)/2}$; in this case, the desired map g can be given by multiplying the argument of each z_i by k without changing the norm of z_i. Then it is clear that such g has degree k, and hence f has even degree $2k$. □

Problem 4.5 (Hatcher, §2.2, Problem 28)**.**

(1) Use the Mayer–Vietoris sequence to compute the homology groups of the space obtained from a torus $S^1 \times S^1$ by attaching a Möbius band via a homeomorphism from the boundary circle of the Möbius band to the circle $S^1 \times \{x_0\}$ in the torus.

(2) Do the same for the space obtained by attaching a Möbius band to $\mathbb{RP}^2$ via a homeomorphism of its boundary circle to the standard $\mathbb{RP}^1 \subset \mathbb{RP}^2$.

Solution. (1) Let $X = T \cup_{S^1} M$ with $T \simeq S^1 \times S^1$ the torus and M the Möbius band. Notice that $\dim X = 2$. The Mayer–Vietoris sequence for X is given as

$$\begin{array}{l} 0 \longrightarrow H_2(T \cap M) \longrightarrow H_2(T) \oplus H_2(M) \longrightarrow H_2(X) \\ \hookrightarrow H_1(T \cap M) \longrightarrow H_1(T) \oplus H_1(M) \longrightarrow H_1(X) \\ \hookrightarrow H_0(T \cap M) \longrightarrow H_0(T) \oplus H_0(M) \longrightarrow H_0(X) \longrightarrow 0. \end{array}$$

To proceed on, we make the following observations.

- Up to deformation retract, we have homotopy equivalences $M \approx S^1$ and $T \cap M \approx S^1$, and hence $H_i(T \cap M) \cong H_i(M) \cong H_i(S^1)$. Recall that $H_i(S^1) \cong \mathbb{Z}$ for $i = 0, 1$ and it vanishes for $i = 2$.
- Referring to Problem 3.1(1) and Problem 3.3 for homology of the torus $T \simeq S^1 \times S^1$, we obtain $H_0(T) \cong H_2(T) \cong \mathbb{Z}$ and $H_1(T) \cong \mathbb{Z} \oplus \mathbb{Z}$.
- Since X is path-connected, we have $H_0(X) \cong \mathbb{Z}$.

Combining the ingredients above, the Mayer–Vietoris sequence for X becomes

$$\begin{array}{l} 0 \longrightarrow 0 \longrightarrow \mathbb{Z} \oplus 0 \longrightarrow H_2(X) \\ \hookrightarrow \mathbb{Z} \xrightarrow{\phi_1} (\mathbb{Z} \oplus \mathbb{Z}) \oplus \mathbb{Z} \longrightarrow H_1(X) \\ \hookrightarrow \mathbb{Z} \xrightarrow{\phi_0} \mathbb{Z} \oplus \mathbb{Z} \longrightarrow \mathbb{Z} \longrightarrow 0. \end{array}$$

Observe that, by construction, the map ϕ_1 above is given by $a \mapsto ((a, 0), 2a)$, and thus ϕ_1 is injective. It follows that $H_2(X) \to H_1(T \cap M) \cong \mathbb{Z}$ must be the zero map. Then we deduce $H_2(X) \cong H_2(T) \oplus H_2(M) \cong \mathbb{Z}$. Similarly, ϕ_0 is injective as well, and hence $H_1(X) \to H_0(T \cap M) \cong \mathbb{Z}$ must be zero. Then the middle row above actually gives rise to a short exact sequence $0 \to \mathbb{Z} \to (\mathbb{Z} \oplus \mathbb{Z}) \oplus \mathbb{Z} \to H_1(X) \to 0$, and it follows that $H_1(X) \cong \operatorname{coker} \phi_1$; again, as ϕ_1 is injective, we have $\operatorname{coker} \phi_1 \cong \mathbb{Z} \oplus \mathbb{Z}$. To conclude,

$$H_n(X) \cong \begin{cases} \mathbb{Z}, & n = 2, \\ \mathbb{Z} \oplus \mathbb{Z}, & n = 1, \\ \mathbb{Z}, & n = 0. \end{cases}$$

This gives the homology of $X = T \cup_{S^1} M$ as desired.

(2) In this case, $X = \mathbb{RP}^2 \cup_{\mathbb{RP}^1} M$. As in part (1), $\mathbb{RP}^2 \cap M$ is homotopic to S^1 by deformation retract as well, and $H_0(X) \cong \mathbb{Z}$ because X is path-connected. For the homology of $\mathbb{RP}^2$, recall that $H_0(\mathbb{RP}^2) \cong \mathbb{Z}$, $H_1(\mathbb{RP}^2) \cong \mathbb{Z}/2\mathbb{Z}$, and $H_n(\mathbb{RP}^2) = 0$ for $n \geqslant 2$. So the Mayer–Vietoris sequence for X becomes

$$\begin{array}{l} 0 \longrightarrow 0 \longrightarrow 0 \oplus 0 \longrightarrow H_2(X) \\ \hookrightarrow \mathbb{Z} \xrightarrow{\phi_1} \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z} \longrightarrow H_1(X) \\ \hookrightarrow \mathbb{Z} \xrightarrow{\phi_0} \mathbb{Z} \oplus \mathbb{Z} \longrightarrow \mathbb{Z} \longrightarrow 0. \end{array}$$

The map ϕ_1 above is given by $a \mapsto (\bar{a}, 2a)$, where $\bar{a}$ is the image of $a \in \mathbb{Z}$ in $\mathbb{Z}/2\mathbb{Z}$. So ϕ_1 is injective, which forces $H_2(X)$ to be 0. Again, as in part (1), $H_1(X) \cong \operatorname{coker} \phi_1$.

Now it suffices to compute this cokernel. Notice that the image of ϕ_1 is generated by $(1, 2)$ in $\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}$, which is the quotient of $\mathbb{Z} \oplus \mathbb{Z}$ by $(2, 0)$. So the desired cokernel is the quotient of $\mathbb{Z} \oplus \mathbb{Z}$ by $(1, 2)$ and $(2, 0)$, containing four elements $(0, 0), (1, 1), (1, 0), (0, 1)$; here $(1, 0) = 2(1, 1)$, $(0, 1) = 3(1, 1)$, and $(0, 0) = 4(1, 1)$, so the group is cyclic of order 4, generated by $(1, 1)$. To conclude, we have

$$H_n(X) \cong \begin{cases} 0, & n = 2, \\ \mathbb{Z}/4\mathbb{Z}, & n = 1, \\ \mathbb{Z}, & n = 0. \end{cases}$$

This gives the homology of $X = \mathbb{RP}^2 \cup_{\mathbb{RP}^1} M$ as desired. □

Problem 4.6 (Hatcher, §2.2, Problem 30(a)(c)(e))**.** For the mapping torus T_f of a map $f\colon X \to X$, a long exact sequence

$$\cdots \longrightarrow H_n(X) \xrightarrow{\mathbb{1} - f_*} H_n(X) \longrightarrow H_n(T_f) \longrightarrow H_{n-1}(X) \longrightarrow \cdots$$

has been constructed in [Hat02, Example 2.48]. Use this to compute the homology of the mapping tori of the following maps.

(1) A reflection $S^2 \to S^2$.
(2) The map $S^1 \times S^1 \to S^1 \times S^1$ that is the identity on one factor and a reflection on the other.
(3) The map $S^1 \times S^1 \to S^1 \times S^1$ that interchanges the two factors and then reflects one of the factors.

Solution. In each case, we always have $\dim X = 2$ and hence $\dim T_f = 3$ by definition. For this dimension reason, the long exact sequence is written as

$$\begin{array}{l} 0 \longrightarrow H_3(T_f) \\ \hookrightarrow H_2(X) \xrightarrow{\phi_2} H_2(X) \longrightarrow H_2(T_f) \\ \hookrightarrow H_1(X) \xrightarrow{\phi_1} H_1(X) \longrightarrow H_1(T_f) \\ \hookrightarrow H_0(X) \xrightarrow{\phi_0} H_0(X) \longrightarrow H_0(T_f) \longrightarrow 0. \end{array}$$

Here each ϕ_i is given by $\mathbb{1} - f_*$. For ϕ_0, this is the zero map by construction, so

$$H_0(T_f) \cong H_0(X).$$

For ϕ_1, the condition $\phi_0 = 0$ gives a short exact sequence $0 \to \operatorname{coker} \phi_1 \to H_1(T_f) \to H_0(X) \to 0$. For ϕ_2, since $H_3(X)$ vanishes we have

$$H_3(T_f) \cong \ker \phi_2.$$

Also there is a short exact sequence $0 \to \operatorname{coker} \phi_2 \to H_2(T_f) \to \ker \phi_1 \to 0$.

For our purpose, we want to compute $H_1(T_f)$ and $H_2(T_f)$ in terms of the kernels and cokernels. So we need the sequences above to split. Viewing all terms as $\mathbb{Z}$-modules, it suffices to show that $H_0(X)$ and $\ker \phi_1$ are torsion-free. Indeed, when X is S^2 or $S^1 \times S^1$ (see Problem 3.1(1) for their homology groups), both $H_0(X)$ and $H_1(X)$ are torsion free, and so also is $\ker\phi_1$ as a submodule of $H_1(X)$. Thus, by Problem 2.6(1), we attain

$$\begin{aligned} H_1(T_f) &\cong \operatorname{coker}\phi_1 \oplus H_0(X),\\ H_2(T_f) &\cong \operatorname{coker}\phi_2 \oplus \ker\phi_1. \end{aligned}$$

Now it remains to compute $\operatorname{coker}\phi_1$, $\operatorname{coker}\phi_2$, and $\ker\phi_1$.

(1) Let $X = S^2$ and f be the reflection map. We have $H_0(X) \cong H_2(X) \cong \mathbb{Z}$ and $H_1(X) = 0$; the latter implies $\ker\phi_1 = \operatorname{coker}\phi_1 = 0$. Also, f_* multiplies each homology class by -1, and hence $\mathbb{1} - f_*$ is the multiplication by 2. In particular, ϕ_2 is injective so that $\ker\phi_2 = 0$, and $\operatorname{coker}\phi_2 \cong \mathbb{Z}/2\mathbb{Z}$. To conclude, in this case we have

$$H_n(T_f) \cong \begin{cases} 0, & n = 3,\\ \mathbb{Z}/2\mathbb{Z}, & n = 2,\\ \mathbb{Z}, & n = 1,\\ \mathbb{Z}, & n = 0. \end{cases}$$

(2) Let $X = S^1 \times S^1$. Then $H_0(X) \cong H_2(X) \cong \mathbb{Z}$ and $H_1(X) \cong \mathbb{Z} \oplus \mathbb{Z}$. The map f is induced by $(x, y) \mapsto (x, -y)$ on $\mathbb{R} \times \mathbb{R}$. So f_* is $\mathbb{1}$ on one S^1 and has the same behavior as in part (1) on the other S^1.

- At the level of 1-cycles, f acts on each component of $S^1 \times S^1$ separately, so $\phi_1 \colon (a, b) \mapsto (a, b) - (a, -b) = (0, 2b)$ on $H_1(X)$.
- At the level of 2-cycles, f acts on X globally, and hence f_* is given by the multiplication by $\det f$; indeed, f_* can be represented by matrix $\left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$ with determinant -1, so $\phi_2 \colon a \mapsto (1 - \det f) \cdot a = 2a$.

Therefore, we have $\ker\phi_1 \cong \mathbb{Z}$ and $\operatorname{coker}\phi_2 \cong \mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$, together with $\ker\phi_2 = 0$ and $\operatorname{coker}\phi_2 \cong \mathbb{Z}/2\mathbb{Z}$. This leads to the conclusion that

$$H_n(T_f) \cong \begin{cases} 0, & n = 3,\\ \mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}, & n = 2,\\ \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}, & n = 1,\\ \mathbb{Z}, & n = 0. \end{cases}$$

(3) Let X be the same as part (2) but f induced by $(x, y) \mapsto (-y, x)$ on $\mathbb{R} \times \mathbb{R}$ instead. Similar to part (2), we make the following argument.

- At the level of 1-cycles, we have $\phi_1 \colon (a, b) \mapsto (a, b) + (-b, a) = (a - b, a + b)$.
- At the level of 2-cycles, notice that f is represented by matrix $\left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$ with determinant 1, so $\phi_2 \colon a \mapsto (1 - \det f) \cdot a = 0$.

It follows that $\ker\phi_1 = 0$ and $\operatorname{coker}\phi_2 \cong \mathbb{Z}/2\mathbb{Z}$, together with $\ker\phi_2 \cong \operatorname{coker}\phi_2 \cong \mathbb{Z}$. As a result,

$$H_n(T_f) \cong \begin{cases} \mathbb{Z}, & n = 3,\\ \mathbb{Z}, & n = 2,\\ \mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}, & n = 1,\\ \mathbb{Z}, & n = 0. \end{cases}$$

□

Homework 5

Problem 5.1 (MIT 18.905, Problem Set V, Problem 19)**.** Verify the following lemma. Let I be a directed set, L an abelian group, and $A\colon I \to \mathsf{Ab}$ an I-directed diagram of abelian groups, with bonding maps $f_{ij}\colon A_i \to A_j$ for $i \leq j$. A map $A \to c_L$, given by compatible maps $f_i\colon A_i \to L$, is a direct limit if and only if

(i) For any $b \in L$ there exists $i \in I$ and $a_i \in A_i$ such that $f_i a_i = b$, and
(ii) For any $a_i \in A_i$ such that $f_i(a_i) = 0 \in L$, there exists j such that $i \leq j$ and $f_{ij}(a_i) = 0 \in A_j$.

Solution. By definition, c_L is the constant complex formed by L, and $A \to c_L$ being a direct limit means that c_L is initial over A. Assume first $A \to c_L$ is a direct limit and check (i) and (ii). Let L_0 be the subgroup of L generated by all images $f_i(A_i)$'s with $i \in I$, then $A \to c_L$ factors through c_{L_0}; on the other hand, the assumption implies that $A \to c_{L_0}$ factors through c_L, so we deduce $L = L_0$. In particular, this implies (i). As for (ii), consider $p_k\colon A \to A_k$ by defining $(p_k)_j = f_{jk}$ for $j \leq k$ and $(p_k)_j = 0$ for $j \not\leq k$. Then each p_k factors through c_L. If $a_i \in A_i$ is eventually nonzero in A, then it has nonzero image along p_i, and hence must be nonzero in c_L. So the condition (ii) follows.

We then assume (i) and (ii) hold and work for the converse direction. Given $g\colon A \to c_K$ for any $K \in \mathsf{Ab}$ in terms of $(g_i\colon A_i \to K)_{i\in I}$, we define $t_{L,K}\colon L \to K,\ l \mapsto g_i(f_i^{-1}(l))$ by choosing for each l a sufficiently large $i \in I$ with respect to the order $\leq$ on I; here $f_i^{-1}(l)$ makes sense because of (i). Using (ii), we see whenever $l = 0$, we can choose i such that $f_i^{-1}(l) = 0$, and then $t_{L,K}(0) = 0$, meaning that this is a group homomorphism. Further, it is clear that g factors as $t_{L,K} \circ f$ uniquely, so it only remains to checking that $t_{L,K}(l)$ is independent of the choice of i. But for $j \neq i$ such that $f_j^{-1}(l)$ exists, if $j \geq i$ then $(g_i f_i^{-1})(l) = (g_j f_{ij} f_i^{-1})(l) = (g_j f_j^{-1})(l)$, so we see $t_{L,K}$ is well-defined. This completes the proof. □

Problem 5.2 (MIT 18.905, Problem Set V, Problem 20)**.**

(1) Embed $\mathbb{Z}/p^n\mathbb{Z}$ into $\mathbb{Z}/p^{n+1}\mathbb{Z}$ by sending 1 to p, and write $\mathbb{Z}_{p^\infty}$ for the union. It is called the Prüfer group at p. Show that $\mathbb{Z}_{p^\infty} \cong \mathbb{Z}[1/p]/\mathbb{Z}$ and that $\mathbb{Q}/\mathbb{Z} \cong \bigoplus_p \mathbb{Z}_{p^\infty}$, where the sum runs over the prime numbers.
(2) Compute $\mathbb{Z}_{p^\infty} \otimes_{\mathbb{Z}} A$ for A each of the following abelian groups: $\mathbb{Z}/n\mathbb{Z}$, $\mathbb{Z}[1/q]$ (for q a prime), and $\mathbb{Z}_{q^\infty}$ (for q a prime).
(3) Compute $\operatorname{Tor}_1^{\mathbb{Z}}(M, \mathbb{Z}[1/p])$ and $\operatorname{Tor}_1^{\mathbb{Z}}(M, \mathbb{Z}_{p^\infty})$, for any abelian group M in terms of the self-map $p\colon M \to M$.

Solution. (1) To show the first isomorphism, notice that each element in $\mathbb{Z}[1/p]/\mathbb{Z}$ is of the form a/p^n for $n \in \mathbb{Z}_{\geqslant 0}$ and $0 \leqslant a < p^n$. Define the map

$$\varphi_n\colon \mathbb{Z}/p^n\mathbb{Z} \longrightarrow \mathbb{Z}[1/p]/\mathbb{Z}, \quad a \longmapsto a/p^n,$$

which is clearly an injective homomorphism of additive groups. On the other hand, each element of the target comes from the image of some φ_k. Taking the limit, we obtain the group isomorphism $\mathbb{Z}_{p^\infty} \cong \mathbb{Z}[1/p]/\mathbb{Z}$.

Using this, showing the second isomorphism is equivalent to showing $\mathbb{Q}/\mathbb{Z} \cong \bigoplus_p \mathbb{Z}[1/p]/\mathbb{Z}$. Indeed, we obtain a naive embedding $\mathbb{Z}[1/p] \hookrightarrow \mathbb{Q}$ for each p, which further gives rise to

$$\psi\colon \textstyle\bigoplus_p \mathbb{Z}[1/p]/\mathbb{Z} \longrightarrow \mathbb{Q}/\mathbb{Z}, \quad (a_p/p^{n_p})_p \longmapsto \sum_p a_p/p^{n_p}.$$

Here the sum over p is finite, i.e. $a_p = 0$ for all but finitely many p's. Clearly, ψ is a homomorphism of additive groups. For injectivity, assume $\sum_p a_p/p^{n_p} \in \mathbb{Z}$ and then p^{n_p} divides the product of a_p with ℓ^{n_ℓ}'s where ℓ runs through all primes $\ell \neq p$, and then $p^{n_p} \mid a_p$ follows. So $\sum_p a_p/p^{n_p} \in \mathbb{Z}$ implies $a_p/p^{n_p} \in \mathbb{Z}$ for each p, and hence ψ is injective. For the surjectivity, it suffices to construct the preimage of elements of the form $\alpha = 1/p_1^{n_1} \cdots p_r^{n_r} \in \mathbb{Q}/\mathbb{Z}$. Notice that $\gcd(p_1^{n_1} \cdots p_r^{n_r}/p_j^{n_j}; 1 \leqslant j \leqslant r) = 1$; by Bézout's theorem, there are $a_1, \ldots, a_r \in \mathbb{Z}$ such that $\sum_{j=1}^r a_j(p_1^{n_1} \cdots p_r^{n_r}/p_j^{n_j}) = 1$, or equivalently $\alpha = \sum_{j=1}^r a_j/p_j^{n_j}$, showing that ψ is surjective.

(2) Note that for $\mathbb{Z}$-modules the tensor product commutes with direct limits, so we have

$$\mathbb{Z}_{p^\infty} \otimes_{\mathbb{Z}} A = \varinjlim_n (\mathbb{Z}/p^n\mathbb{Z} \otimes_{\mathbb{Z}} A).$$

We are using this observation to do the following computations.

(i) When $A = \mathbb{Z}/n\mathbb{Z}$, write $n = p^k m$ for $p \nmid m$. Since $\mathbb{Z}/p^n\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/m\mathbb{Z} = 0$ for each n, we have $\mathbb{Z}_{p^\infty} \otimes_{\mathbb{Z}} \mathbb{Z}/m\mathbb{Z} = 0$ by the observation above. It reduces to the case where $m = 1$, or equivalently $n = p^k$ and $A = \mathbb{Z}/p^k\mathbb{Z}$. By part (1), $\mathbb{Z}_{p^\infty} \otimes_{\mathbb{Z}} A \cong \mathbb{Z}[1/p]/\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/p^k\mathbb{Z}$; but the tensor product of the p-divisible and the p-torsion must be 0, so we conclude that $\mathbb{Z}_{p^\infty} \otimes_{\mathbb{Z}} A = 0$.

(ii) When $A = \mathbb{Z}[1/q]$, we need to compute $\mathbb{Z}_{p^\infty} \otimes_{\mathbb{Z}} A \cong (\mathbb{Z}[1/p]/\mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{Z}[1/q]$ by (1) again. Since tensor product is right exact (i.e. commutes with quotient) and $\mathbb{Z}$ is torsion-free, this equals $(\mathbb{Z}[1/p] \otimes_{\mathbb{Z}} \mathbb{Z}[1/q])/\mathbb{Z}[1/q] = \mathbb{Z}[1/p, 1/q]/\mathbb{Z}[1/q]$.
   - If $p = q$, the desired result is 0.
   - If $p \neq q$, notice that $\mathbb{Z}[1/p, 1/q]/\mathbb{Z} \cong \mathbb{Z}[1/p]/\mathbb{Z} \oplus \mathbb{Z}[1/q]/\mathbb{Z}$, where the second summand is trivialized modulo $\mathbb{Z}[1/q]$. It follows that $\mathbb{Z}[1/p, 1/q]/\mathbb{Z}[1/q] \cong \mathbb{Z}[1/p]/\mathbb{Z} \cong \mathbb{Z}_{p^\infty}$.

(iii) When $A = \mathbb{Z}_{q^\infty}$, by the observation before, $\mathbb{Z}_{p^\infty} \otimes_{\mathbb{Z}} A = \varinjlim_{m,n} (\mathbb{Z}/p^n \otimes_{\mathbb{Z}} \mathbb{Z}/q^m)$. The two cases are as follows.
   - If $p = q$, at a finite level $\mathbb{Z}_{p^\infty} \otimes_{\mathbb{Z}} \mathbb{Z}/p\mathbb{Z} = 0$ by (i), so $\mathbb{Z}_{p^\infty} \otimes_{\mathbb{Z}} \mathbb{Z}_{p^\infty} = 0$ by the observation at the beginning.
   - If $p \neq q$, at a finite level $\mathbb{Z}/p\mathbb{Z} \otimes \mathbb{Z}/q\mathbb{Z} = 0$, so $\mathbb{Z}_{p^\infty} \otimes_{\mathbb{Z}} \mathbb{Z}_{q^\infty} = 0$ by the observation at the beginning.

To sum up, for A being either of $\mathbb{Z}/n\mathbb{Z}$, $\mathbb{Z}[1/q]$, or $\mathbb{Z}_{q^\infty}$, we have the following:

$$\mathbb{Z}_{p^\infty} \otimes_{\mathbb{Z}} A = \begin{cases} \mathbb{Z}_{p^\infty}, & A = \mathbb{Z}[1/p], \\ 0, & \text{else.} \end{cases}$$

(3) Notice that $\mathbb{Z}[1/p]$ is a flat $\mathbb{Z}$-module as a localization of $\mathbb{Z}$. Equivalently, it means that a projective resolution of $\mathbb{Z}[1/p]$ can be obtained by localizing $0 \to \mathbb{Z} \xrightarrow{p} \mathbb{Z} \to \mathbb{Z}/p\mathbb{Z} \to 0$ at p, in whose result the multiplication-by-p map becomes invertible, and hence has a trivial kernel. So we always have

$$\operatorname{Tor}_1^{\mathbb{Z}}(M, \mathbb{Z}[1/p]) = 0.$$

We also compute $\operatorname{Tor}_1^{\mathbb{Z}}(M, \mathbb{Z}_{p^\infty})$. By definition, for each n,

$$\operatorname{Tor}_1(M, \mathbb{Z}/p^n\mathbb{Z}) \cong \ker(p^n \colon M \to M) =: M[p^n].$$

Taking direct limit, it follows that

$$\operatorname{Tor}_1(M, \mathbb{Z}_{p^\infty}) \cong \varinjlim_n M[p^n] = M[p^\infty].$$

This also equals the union of $\ker(p^n \colon M \to M)$ for all n. □

Problem 5.3 (MIT 18.905, Problem Set V, Problem 21)**.** Show that if $f \colon X \to Y$ induces an isomorphism in homology with coefficients in the prime fields $\mathbb{F}_p$ (for all primes p) and $\mathbb{Q}$, then it induces an isomorphism in homology with coefficients in $\mathbb{Z}$.

Solution. By Problem 5.2(1), the isomorphism $\mathbb{Q}/\mathbb{Z} \cong \bigoplus_p \mathbb{Z}_{p^\infty}$ induces a short exact sequence

$$0 \longrightarrow \mathbb{Z} \longrightarrow \mathbb{Q} \longrightarrow \bigoplus_p \mathbb{Z}_{p^\infty} \longrightarrow 0.$$

Taking homology on this and applying the condition $H_i(X;\mathbb{Q}) \cong H_i(Y;\mathbb{Q})$, for the purpose that $H_i(X;\mathbb{Z}) \cong H_i(Y;\mathbb{Z})$, it suffices to show $H_i(X;\mathbb{Z}_{p^\infty}) \cong H_i(Y;\mathbb{Z}_{p^\infty})$ for each prime p. On the other hand, for each $k \geqslant 1$ there is a short exact sequence

$$0 \longrightarrow \mathbb{F}_{p^k} \longrightarrow \mathbb{F}_{p^{k+1}} \longrightarrow \mathbb{F}_p \longrightarrow 0.$$

So by induction, the condition that $H_i(X;\mathbb{F}_p) \cong H_i(Y;\mathbb{F}_p)$ implies $H_i(X;\mathbb{F}_{p^k}) \cong H_i(Y;\mathbb{F}_{p^k})$ for each k. Thus, taking direct limit, it follows that $H_i(X;\mathbb{Z}_{p^\infty}) \cong H_i(Y;\mathbb{Z}_{p^\infty})$. This completes the proof. □

Alternative Solution. This solution makes use of the mapping cone of f to measure the failure of f to be an isomorphism at homological level. Denote C_f the mapping cone of $f \colon X \to Y$, satisfying the long exact sequence

$$\cdots \longrightarrow H_n(X;\Lambda) \xrightarrow{f_*} H_n(Y;\Lambda) \longrightarrow H_n(\mathrm{C}_f;\Lambda) \longrightarrow H_{n-1}(X;\Lambda) \longrightarrow \cdots$$

For our purpose, we only need to show $H_n(\mathrm{C}_f) = H_n(\mathrm{C}_f;\mathbb{Z}) = 0$. By assumption, f_* induces isomorphisms on each n for $\Lambda = \mathbb{F}_p$ and $\mathbb{Q}$. Then, applying the universal coefficient theorem to the case $\Lambda = \mathbb{F}_p$, we obtain

$$H_n(\mathrm{C}_f) \otimes_{\mathbb{Z}} \mathbb{F}_p = \mathrm{Tor}_1(H_{n-1}(\mathrm{C}_f), \mathbb{F}_p) = 0,$$

and it follows that $H_n(\mathrm{C}_f) = H_n(\mathrm{C}_f) \otimes_{\mathbb{Z}} \mathbb{Z}$ is p-torsion-free for any prime p. On the other hand, for the free part of $H_n(\mathrm{C}_f)$, the universal coefficient theorem for $\Lambda = \mathbb{Q}$ implies

$$H_n(\mathrm{C}_f) \otimes_{\mathbb{Z}} \mathbb{Q} = \mathrm{Tor}_1(H_{n-1}(\mathrm{C}_f), \mathbb{Q}) = 0.$$

In fact, the vanishing of the second group always holds unconditionally as $\mathbb{Q}$ is flat. Since $H_n(\mathrm{C}_f)$ is torsion-free, the vanishing of $H_n(\mathrm{C}_f) \otimes_{\mathbb{Z}} \mathbb{Q}$ renders that $H_n(\mathrm{C}_f) = 0$. As a result, we attain $H_n(X;\mathbb{Z}) \cong H_n(Y;\mathbb{Z})$ as desired. ☐

Problem 5.4 (MIT 18.905, Problem Set V, Problem 22)**.** The construction of an isomorphism between the singular homology and the cellular homology of a CW complex carries over *verbatim* with any coefficients. Use this observation to compute the homology of $\mathbb{RP}^n$ with coefficients in $\mathbb{Q}$, $\mathbb{F}_p$, $\mathbb{Z}_{p^\infty}$, and $\mathbb{Z}[1/p]$.

Solution. At each dimension the CW structure of $\mathbb{RP}^n$ contains two antipodal cells from the quotient of that in S^n. At degree n, the boundary map is given by multiplication by $1 + \deg(-\mathbb{1})$, where $-\mathbb{1}\colon S^n \to S^n$ is the antipodal map (see Problem 4.2 for an example). So the cellular complex of $\mathbb{RP}^n$ is read as

$$0 \longrightarrow \mathbb{Z} \longrightarrow \cdots \longrightarrow \mathbb{Z} \xrightarrow{\times 0} \mathbb{Z} \xrightarrow{\times 2} \mathbb{Z} \xrightarrow{\times 0} \mathbb{Z} \longrightarrow 0,$$

where from the left the first $\mathbb{Z}$ lies in degree n and the last $\mathbb{Z}$ lies in degree 0. Changing the coefficient in a verbatim sense, for $\Lambda \in \{\mathbb{Q}, \mathbb{F}_p, \mathbb{Z}_{p^\infty}, \mathbb{Z}[1/p]\}$ we obtain

$$0 \longrightarrow \Lambda \longrightarrow \cdots \longrightarrow \Lambda \xrightarrow{\times 0} \Lambda \xrightarrow{\times 2} \Lambda \xrightarrow{\times 0} \Lambda \longrightarrow 0.$$

We then compute $H_i(\mathbb{RP}^n;\Lambda)$ as follows.

(i) When $\Lambda = \mathbb{Q}$, the multiplication-by-2 map is invertible, and hence induces automorphisms on Λ at odd degrees. So $H_i(\mathbb{RP}^n;\mathbb{Q}) = \mathbb{Q}$ for $i = 0$ and vanishes for $0 < i < n$. For $i = n$, it is $\mathbb{Q}$ when n is odd and vanishes when n is even.

(ii) When $\Lambda = \mathbb{F}_p$, the multiplication-by-2 map is invertible for $p \neq 2$, in which case each $H_i(\mathbb{RP}^n;\Lambda)$ is the same as in (i) with $\mathbb{Q}$ replaced by $\mathbb{F}_p$. Whereas for $p = 2$ all boundary maps become zero and we obtain $H_i(\mathbb{RP}^n;\mathbb{F}_p) = \mathbb{F}_2$ for all $0 \leqslant i \leqslant n$.

(iii) When $\Lambda = \mathbb{Z}_{p^\infty}$, the multiplication-by-2 map is an isomorphism for $p \neq 2$ as it is invertible on each $\mathbb{Z}/p^m\mathbb{Z}$ at a finite level (c.f. Problem 5.2(1)). So each $H_i(\mathbb{RP}^n;\Lambda)$ is the same as in (i) with $\mathbb{Q}$ replaced by $\mathbb{Z}_{p^\infty}$. As for $p = 2$, the multiplication-by-2 map is surjective with kernel $\Lambda/2\Lambda \cong \mathbb{Z}/2\mathbb{Z}$. So for $0 < i \leqslant n$ with i even, $H_i(\mathbb{RP}^n;\mathbb{Z}_{p^\infty}) \cong \mathbb{Z}/2\mathbb{Z}$.

(iv) When $\Lambda = \mathbb{Z}[1/p]$, the multiplication-by-2 map is invertible if only if $p = 2$; in this case, each $H_i(\mathbb{RP}^n;\Lambda)$ is the same as in (i) with $\mathbb{Q}$ replaced by $\mathbb{Z}[1/p]$. As for $p \neq 2$, inverting p does not change anything compared to the case of $\mathbb{Z}$; in this case, the results for $i = 0, n$ are the same as before, but for $0 < i < n$ with i odd, $H_i(\mathbb{RP}^n;\mathbb{Z}[1/p]) = \Lambda/2\Lambda \cong \mathbb{Z}/2\mathbb{Z}$.

To sum up, for $\Lambda \in \{\mathbb{Q}, \mathbb{F}_p, \mathbb{Z}_{p^\infty}, \mathbb{Z}[1/2] \colon p \neq 2\}$, we conclude that

$$H_i(\mathbb{RP}^n;\Lambda) = \begin{cases} \Lambda, & i = 0 \text{ or } n \text{ for } n \text{ odd}, \\ 0, & \text{else}. \end{cases}$$

For other cases, we also obtain

$$H_i(\mathbb{RP}^n;\mathbb{F}_2) = \begin{cases} \mathbb{F}_2, & 0 \leqslant i \leqslant n, \\ 0, & \text{else}; \end{cases}$$

and

$$H_i(\mathbb{RP}^n;\mathbb{Z}_{2^\infty}) = \begin{cases} \mathbb{Z}_{2^\infty}, & i = 0 \text{ or } n \text{ for } n \text{ odd}, \\ \mathbb{Z}/2\mathbb{Z}, & 0 < i \leqslant n \text{ even}, \\ 0, & \text{else}. \end{cases}$$

Also, with $p \neq 2$,

$$H_i(\mathbb{RP}^n; \mathbb{Z}[1/p]) = \begin{cases} \mathbb{Z}[1/p], & i = 0 \text{ or } n \text{ for } n \text{ odd}, \\ \mathbb{Z}/2\mathbb{Z}, & 0 < i < n \text{ odd}, \\ 0, & \text{else}. \end{cases}$$

These form the whole list of $H_i(\mathbb{RP}^n; \Lambda)$ in all cases. □

Problem 5.5 (MIT 18.905, Problem Set V, Problem 24)**.** Suppose that $f\colon X \to Y$ induces an isomorphism in homology with coefficients in $\mathbb{Z}/n\mathbb{Z}$. Show that it induces an isomorphism in homology with coefficients in any abelian group in which every element is killed by some power of n.

Solution. Similar to the construction of $\mathbb{Z}_{p^\infty}$, define $\mathbb{Z}_{n^\infty}$ to be the Prüfer n-group containing all n-torsions. We need to show that

$$H_i(X; \mathbb{Z}_{n^\infty}) \cong H_i(Y; \mathbb{Z}_{n^\infty}).$$

Using the result in Problem 5.2(1), we have $\mathbb{Z}_{n^\infty} \cong \mathbb{Z}[1/n]/\mathbb{Z} \cong \bigoplus_{p|n} \mathbb{Z}[1/p]/\mathbb{Z} \cong \bigoplus_{p|n} \mathbb{Z}_{p^\infty}$. So it reduces to the case that $n = p$ being a prime. It further reduces to the finite level, i.e., it suffices to show $H_i(X; \mathbb{Z}/p^k\mathbb{Z}) \cong H_i(Y; \mathbb{Z}/p^k\mathbb{Z})$ for each $k \in \mathbb{Z}_{\geqslant 1}$.

Indeed, this can be proved by using a similar argument in the first solution of Problem 5.3. Applying induction on k, the case $k = 1$ is clear by assumption. Assume the isomorphism holds for some $k \geqslant 2$. Then there is an exact sequence

$$0 \longrightarrow \mathbb{Z}/n^k\mathbb{Z} \longrightarrow \mathbb{Z}/n^{k+1}\mathbb{Z} \longrightarrow \mathbb{Z}/n\mathbb{Z} \longrightarrow 0.$$

Taking homology on this, we see for each i, the isomorphisms $H_i(X; \mathbb{Z}/n^k\mathbb{Z}) \cong H_i(Y; \mathbb{Z}/n^k\mathbb{Z})$ together with $H_i(X; \mathbb{Z}/n\mathbb{Z}) \cong H_i(Y; \mathbb{Z}/n\mathbb{Z})$ imply $H_i(X; \mathbb{Z}/n^{k+1}\mathbb{Z}) \cong H_i(Y; \mathbb{Z}/n^{k+1}\mathbb{Z})$. So the desired result follows. □

Homework 6

Problem 6.1 (Hatcher, §3.A, Problem 1)**.** Use the universal coefficient theorem to show that if $H_*(X;\mathbb{Z})$ is finitely generated, so the Euler characteristic $\chi(X) = \sum_n (-1)^n \operatorname{rank}_{\mathbb{Z}} H_n(X;\mathbb{Z})$ is defined, then for any coefficient field F we have $\chi(X) = \sum_n (-1)^n \dim_F H_n(X;F)$.

Solution. Using universal coefficient theorem, we see

$$\dim_F H_n(X;F) = \dim_F(H_n(X;\mathbb{Z}) \otimes_{\mathbb{Z}} F) + \dim_F \operatorname{Tor}(H_{n-1}(X;\mathbb{Z}), F).$$

But this does not peel off all the torsions of $H_n(X;F)$, so one will need to consider the torsions inside $H_n(X;\mathbb{Z}) \otimes_{\mathbb{Z}} F$. As $H_n(X;\mathbb{Z})$ is finitely generated, there is an isomorphism $H_n(X;\mathbb{Z}) \cong \mathbb{Z}^{r_n} \oplus T_n$ with $r_n = \operatorname{rank}_{\mathbb{Z}} H_n(X;\mathbb{Z})$ and T_n being the torsion part, so that $H_n(X;\mathbb{Z}) \otimes_{\mathbb{Z}} F \cong F^{r_n} \oplus (T_n \otimes_{\mathbb{Z}} F)$. Taking F-dimensions on this isomorphism between vector spaces, we get

$$\begin{aligned}\dim_F(H_n(X;\mathbb{Z}) \otimes_{\mathbb{Z}} F) &= \dim_F F^{r_n} + \dim_F(T_n \otimes_{\mathbb{Z}} F) \\ &= \operatorname{rank}_{\mathbb{Z}} H_n(X;\mathbb{Z}) + \dim_F(T_n \otimes_{\mathbb{Z}} F).\end{aligned}$$

Plugging the formula above in to the formula of $\dim_F H_n(X;F)$, for our purpose it suffices to show

$$\sum_n (-1)^n \dim_F(T_n \otimes_{\mathbb{Z}} F) + \sum_n (-1)^n \dim_F \operatorname{Tor}(H_{n-1}(X;\mathbb{Z}), F) = 0.$$

For this, by construction, $\operatorname{Tor}(H_{n-1}(X;\mathbb{Z}), F) = \operatorname{Tor}(T_{n-1}, F) \cong T_{n-1} \otimes_{\mathbb{Z}} F$. So the desired equality is equivalent to

$$\sum_n (-1)^n (\dim_F(T_n \otimes_{\mathbb{Z}} F) + \dim_F(T_{n-1} \otimes_{\mathbb{Z}} F)) = 0.$$

Its left hand side can be further rewritten as

$$\sum_n (-1)^n \dim_F(T_n \otimes_{\mathbb{Z}} F) - \sum_n (-1)^n \dim_F(T_{n-1} \otimes_{\mathbb{Z}} F) = \dim_F(T_0 \otimes_{\mathbb{Z}} F).$$

But the right hand side equals zero as $H_0(X;\mathbb{Z})$ is always torsion-free. □

Problem 6.2 (Hatcher, §3.A, Problem 2)**.** Show that $\operatorname{Tor}(A, \mathbb{Q}/\mathbb{Z})$ is isomorphic to the torsion subgroup of A. Deduce that A is torsion-free iff $\operatorname{Tor}(A, B) = 0$ for all B.

Solution. By Problem 5.2(1)(3), we have

$$\operatorname{Tor}(A, \mathbb{Q}/\mathbb{Z}) \cong \operatorname{Tor}(A, \textstyle\bigoplus_p \mathbb{Z}_{p^\infty}) \cong \bigoplus_p \operatorname{Tor}(A, \mathbb{Z}_{p^\infty}) \cong \bigoplus_p A[p^\infty].$$

Recall that $A[p^\infty]$ is the union of all $\ker(p^n\colon A \to A)$ for $n \geqslant 1$, so every element in $\bigoplus_p A[p^\infty]$ is torsion. In particular, $\operatorname{Tor}(A, \mathbb{Q}/\mathbb{Z})$ is a torsion subgroup of A. Conversely, using the argument in Problem 5.2(1) that shows $\bigoplus_p \mathbb{Z}[1/p]\mathbb{Z} \cong \mathbb{Q}/\mathbb{Z}$, we deduce that $\bigoplus_p A[p^\infty]$ covers all torsion elements in A. This proves the first assertion.

Next, assume A is torsion-free as a $\mathbb{Z}$-module. Then there is a free $\mathbb{Z}$-module F together with an injection $A \hookrightarrow F$ (this implicitly uses that $\mathbb{Z}$ is a PID and holds even when A is not finitely generated). Since F is clearly flat, so also is its submodule A. It follows that $\operatorname{Tor}(A, B) = 0$ for all B. Conversely, assume that $\operatorname{Tor}(A, B) = 0$ for all B. Then in particular $\operatorname{Tor}(A, \mathbb{Q}/\mathbb{Z}) = 0$, which means A is torsion-free by the first assertion above. □

Problem 6.3 (Hatcher, §3.A, Problem 6)**.** Show that $\operatorname{Tor}(A, B)$ is always a torsion group, and that $\operatorname{Tor}(A, B)$ contains an element of order n iff both A and B contain elements of order n.

Solution. Since $\operatorname{Tor}(-,-)$ commutes with direct limits, it suffices to work for all finitely generated subgroups of A, B, and hence we may assume both A and B are finitely generated.

We first show that $\operatorname{Tor}(A, B)$ is a torsion group. Suppose $\operatorname{Tor}(A, B) \neq 0$, so both A and B are not torsion-free. Since the maximal free subgroups of A and B do not contribute to $\operatorname{Tor}(A, B)$, we may further assume $A \cong \mathbb{Z}/a\mathbb{Z}$ and $B \cong \mathbb{Z}/b\mathbb{Z}$ for simplicity. But in this case $\operatorname{Tor}(A, B) \cong \ker(b\colon A \to A) \cong \mathbb{Z}/\gcd(a,b)\mathbb{Z}$ is always torsion.

For the second assertion, assume both A and B contain elements of order n simultaneously. Then $\mathbb{Z}/n\mathbb{Z}$ appears as normal subgroups of A and B simultaneously, and hence appears in $\operatorname{Tor}(A, B)$. This means $\operatorname{Tor}(A, B)$ contains an element of order n. Conversely, if $\operatorname{Tor}(A, B)$ contains an element of order

n, we may assume A has a normal subgroup of order n and show that so B does. Indeed, any $\mathbb{Z}/m\mathbb{Z}$ appearing in B leads to a direct component $\operatorname{Tor}(\mathbb{Z}/n\mathbb{Z}, \mathbb{Z}/m\mathbb{Z}) \cong \mathbb{Z}/\gcd(n,m)\mathbb{Z}$ of $\operatorname{Tor}(A,B)$; among all these components some $\mathbb{Z}/\gcd(n,m)\mathbb{Z}$ must contain an element of order n, implying that $n \mid m$; so there is an element of order n in B as well. ☐

Problem 6.4 (Hatcher, §3.B, part of Problem 1)**.** Compute the homology group $H_i(\mathbb{RP}^m \times \mathbb{RP}^n; G)$ for $G = \mathbb{Z}$ and $\mathbb{Z}/2\mathbb{Z}$ via the cellular chain complexes (see [Hat02, Example 3B.4]).

Solution. We divide the computation for the desired homology into the following steps.

Step I (Computing the boundary maps)**.** Recall the following from Problem 5.4. In $\mathbb{RP}^k$, there is exactly one i-cell for each $0 \leqslant i \leqslant k$, denoted by e^i. For coefficient ring Λ, this e^i generates $C_i(\mathbb{RP}^k; \Lambda) \cong \Lambda$. When $0 < i \leqslant k$, the boundary map is given by

$$\partial_i \colon C_i(\mathbb{RP}^k; \Lambda) \longrightarrow C_{i-1}(\mathbb{RP}^k; \Lambda), \quad e^i \longmapsto \begin{cases} 0, & 0 < i \leqslant k \text{ odd}, \\ 2e^{i-1}, & 0 < i \leqslant k \text{ even}. \end{cases}$$

The definition of ∂_i also extends to $i = 0$, for which $\partial_0 = 0$. For convenience, we formally write $e^i := 0$ for $i \notin [0,k]$ as a convention in the following.

Consider the space $X := \mathbb{RP}^m \times \mathbb{RP}^n$ with the product CW structure. Denote $e^{(i,j)} := e^i \otimes e^j$ the cell of dimension $i+j$ in X defined by the product of e^i and e^j. Then $C_\mu(X; \Lambda)$ is generated by those $e^{(i,j)}$'s with $i + j = \mu$ for each $0 \leqslant \mu \leqslant m+n$, so we have

$$C_\mu(X; \Lambda) \cong \Lambda^{\oplus r_\mu}, \qquad r_\mu := \#\{(i,j) \in [0,m] \times [0,n] \colon i + j = \mu\}.$$

Using Leibniz's rule, for $0 \leqslant \mu \leqslant m+n$, the μ-th boundary map on X is written as

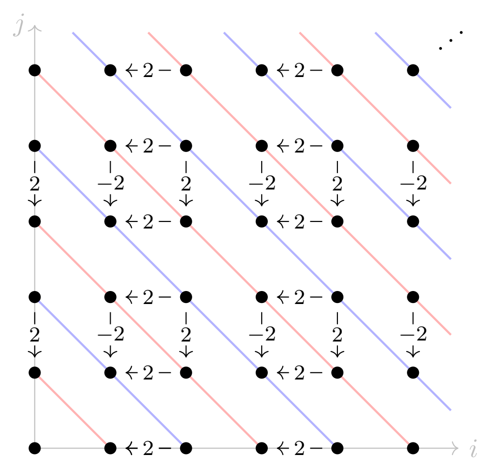
$$\boldsymbol{\partial}_\mu \colon C_\mu(X; \Lambda) \longrightarrow C_{\mu-1}(X; \Lambda), \quad e^{(i,j)} \longmapsto \partial_i(e^i) \otimes e^j + (-1)^i e^i \otimes \partial_j(e_j).$$

This formula works for all $(i,j) \in [0,m] \times [0,n]$ under the prescribed convention.

Step II (Illustration of the boundary maps)**.** Unwinding the formula of $\boldsymbol{\partial}_\mu$, we see

$$\boldsymbol{\partial}_\mu(e^{(i,j)}) = \begin{cases} -2e^{(i,j-1)}, & i \text{ odd and } j \text{ even}, \\ 0, & i \text{ odd and } j \text{ odd}, \\ 2e^{(i-1,j)}, & i \text{ even and } j \text{ odd}, \\ 2e^{(i-1,j)} + 2e^{(i,j-1)}, & i \text{ even and } j \text{ even}. \end{cases}$$

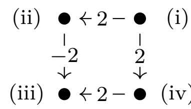
Using the diagrammatic convention in [Hat02, Example 3B.4], this computation deduces the following picture.



In the picture, each node denotes the direct summand of $C_{i+j}(X; \Lambda)$ generated by $e^{(i,j)}$, so each node corresponds to a rank-one Λ-module. The red (resp. blue) lines cascade $e^{(i,j)}$'s with $i+j$ odd (resp. even). For convenience in the next step, we classify all nodes in the picture into either of the following four types:

(i) $e^{(i,j)}$'s with both i, j even;
(ii) $e^{(i,j)}$'s with i odd and j even;
(iii) $e^{(i,j)}$'s with both i, j odd;
(iv) $e^{(i,j)}$'s with i even and j odd.

Thus, for a closed square with ± 2 maps between cells, its four corners exactly correspond to all the four types of nodes.



Step III (Computing the homology)**.** First consider the case $\Lambda = \mathbb{Z}/2\mathbb{Z}$. In this case, all boundary maps $\boldsymbol{\partial}_\mu$ are zero because in the picture $-2 = 2 = 0$. So we obtain

$$H_\mu(X; \mathbb{Z}/2\mathbb{Z}) \cong C_\mu(X; \mathbb{Z}/2\mathbb{Z}) \cong (\mathbb{Z}/2\mathbb{Z})^{\oplus r_\mu}.$$

In the following, we focus on the case $\Lambda = \mathbb{Z}$. Fix $0 \leqslant \mu \leqslant m + n$ and write $\mu = i + j$ for $i, j \in \mathbb{Z}_{\geqslant 0}$. For a cell $e^{(i,j)}$, in order to measure its homological contribution to $H_\mu(X; \Lambda)$, we make the following observations according to the picture above.

- ◇ If $e^{(i,j)}$ is of type (i), it does not lie in $\ker \boldsymbol{\partial}_\mu$ unless $\mu = 0$, and it contains no nontrivial image of $\boldsymbol{\partial}_{\mu+1}$. Therefore, when $\mu \geqslant 1$, its class does not appear in any homology. In contrast, when $\mu = 0$, by our convention $\boldsymbol{\partial}_0 = 0$ and hence $\ker \boldsymbol{\partial}_0 \cong \Lambda$. The latter case corresponds to $(i, j) = (0, 0)$.
- ◇ If $e^{(i,j)}$ is of type (ii) (resp. type (iv)), it pairs up with $e^{(i+1,j-1)}$ of type (iv) (resp. $e^{(i-1,j+1)}$ of type (ii)) if the latter exists. In case of the pairing exists, a rank-one submodule of $\Lambda^{\oplus 2}$ generated by $(1, 1)$ contributes to $\ker \boldsymbol{\partial}_\mu$, where the generator corresponds to $e^{(i,j)} + e^{(i+1,j-1)}$ (resp. $e^{(i,j)} + e^{(i-1,j+1)}$). On the other hand, $\operatorname{im} \boldsymbol{\partial}_{\mu+1}$ is generated by $(2, 2)$. In contrast, if the latter does not exist, we must have $i = m$ (resp. $j = n$), in which case $e^{(i,j)}$ lies in $\ker \boldsymbol{\partial}_\mu$.
- ◇ If $e^{(i,j)}$ is of type (iii), it always vanishes along $\boldsymbol{\partial}_\mu$. This cell also contains $\operatorname{im} \boldsymbol{\partial}_{\mu+1}$, but such image is zero unless any of the following two exceptional cases happens: Exactly one of $e^{(i+1,j)}$ and $e^{(i,j+1)}$ does not exist (which happens when either $i = m$ or $j = n$ but $(i, j) \neq (m, n)$), or alternatively, both $e^{(i+1,j)}$ and $e^{(i,j+1)}$ do not exist (which happens when $(i, j) = (m, n)$). In the first exceptional case, the image of $\boldsymbol{\partial}_{\mu+1}$ is generated by $\pm 2e^{(i,j)}$. In the second exceptional case, we have $\ker \boldsymbol{\partial}_\mu = \ker \boldsymbol{\partial}_{m+n} \cong \Lambda$.

In summary, the structure of $H_\mu(X; \Lambda)$ for $\Lambda = \mathbb{Z}$ is given as follows.

- ◦ Suppose μ is odd (corresponding to the read lines in the picture). The only cells contributing to $H_\mu(X; \mathbb{Z})$ are of type (ii) and (iv). If $e^{(i,j)}$ is on the boundary but not the corners of the picture, it contributes $\mathbb{Z}/2\mathbb{Z}$. If a cell of type (ii) (resp. type (iv)) lies in the lower right corner (resp. upper left corner) of the picture, it contributes $\mathbb{Z}$. Otherwise, a cell of type (ii) and a cell of type (iv) pairs up and they together contribute $\mathbb{Z}/2\mathbb{Z}$; this contribution is isomorphic to the quotient by $(2, 2)$ of the submodule inside $\mathbb{Z}^{\oplus 2}$ generated by $(1, 1)$.
- ◦ Suppose μ is even (corresponding to the blue lines in the picture). We have seen cells of type (i) always make no homological contribution. So the only cells contributing to $H_\mu(X; \mathbb{Z})$ must be of type (iii). A cell of type (iii) cannot be on the boundary of the picture unless it lies in the upper right corner. In the latter case happens the contribution of $e^{(m,n)}$ is $\mathbb{Z}$, but ordinarily the contribution is $\mathbb{Z}/2\mathbb{Z}$ (no matter $e^{(i,j)}$ is on the boundary off the corners in the picture or not).

Step IV (Conclusion)**.** Using the argument in the previous step, we conclude that

$$H_\mu(X; \Lambda) = \begin{cases} \Lambda^{\oplus r_\mu}, & \Lambda = \mathbb{Z}/2\mathbb{Z}, \\ \Lambda^{\oplus f_\mu} \oplus (\Lambda/2\Lambda)^{\oplus g_\mu}, & \Lambda = \mathbb{Z}. \end{cases}$$

Here the integers determined by μ are defined as follows.

- r_μ is the number of pairs of integers $(i, j) \in [0, m] \times [0, n]$ such that $i + j = \mu$.

- g_μ depends on the parity of μ. When μ is odd, g_μ is the number of pairs of nodes of type (ii) and (iv) along the red line $i+j=\mu$. When $\mu \neq m+n$ is even, g_μ is the number of nodes of type (iii) along the blue line $i+j=\mu$.
- $f_\mu = 0$ unless the following cases. We have $f_0 = 1$ for $\mu = 0$; $f_{m+n} = 1$ for both m and n being odd; $f_m = f_n = 2$ for $m = n$ odd; $f_m = 1$ for $m \neq n$ and m odd; $f_n = 1$ for $m \neq n$ and n odd. Equivalently, f_μ is the number of pairs of integers $(i,j) \in \{0,m\} \times \{0,n\}$ such that each of i and j are either 0 or odd.

As a remark, we have only computed the homology in this problem. But one can attain the cohomology by using a duality $H^\mu(X;\Lambda) \cong H_{\dim X - \mu}(X;\Lambda) = H_{m+n-\mu}(X;\Lambda)$. □

Problem 6.5 (Hatcher, §3.B, part of Problem 3)**.** Show that the splitting in the topological Künneth formula cannot be natural by considering the map

$$f \times \mathbb{1} \colon M(\mathbb{Z}/m\mathbb{Z}, n) \times M(\mathbb{Z}/m\mathbb{Z}, n) \longrightarrow S^{n+1} \times M(\mathbb{Z}/m\mathbb{Z}, n)$$

where f collapses the n-skeleton of $M(\mathbb{Z}/m\mathbb{Z}, n) = S^n \cup e^{n+1}$ to a point.

Solution. Recall that the Moore space $M \coloneqq M(\mathbb{Z}/m\mathbb{Z}, n)$ is constructed by attaching an $(n+1)$-cell e^{n+1} of degree m to S^n; in particular, M has dimension $n+1$. Referring to [Hat02, Example 2.40, 2.51], the homology of M is only supported on degrees 0 and n, with $H_0(M) \cong \mathbb{Z}$ and $H_n(M) \cong \mathbb{Z}/m\mathbb{Z}$ (which can be verified through the CW complex). Also recall that the homology of S^{n+1} is only supported on degrees 0 and $n+1$, with $H_0(S^{n+1}) \cong H_{n+1}(S^{n+1}) \cong \mathbb{Z}$.

By construction, f is a homotopy equivalence, so $f \times \mathbb{1}$ induces isomorphisms on homologies at all degrees. On the other hand, at degree $2n+1$, Künneth formula deduces split short exact sequences

$$0 \longrightarrow 0 \longrightarrow H_{2n+1}(M \times M) \longrightarrow \operatorname{Tor}_1^{\mathbb{Z}}(H_n(M), H_n(M)) \longrightarrow 0$$

together with

$$0 \longrightarrow H_{n+1}(S^{n+1}) \otimes H_n(M) \longrightarrow H_{2n+1}(S^{n+1} \times M) \longrightarrow 0 \longrightarrow 0.$$

As $\mathbb{Z}$-modules, both $\operatorname{Tor}_1^{\mathbb{Z}}(H_n(M), H_n(M))$ and $H_{n+1}(S^{n+1}) \otimes H_n(M)$ are isomorphic to $\mathbb{Z}/m\mathbb{Z}$, so one can always choose an isomorphism between $H_{2n+1}(M \times M)$ and $H_{2n+1}(S^{n+1} \times M)$. However, this choice is not canonical, and it at least depends on the choice of a generator ι of $H_{n+1}(S^{n+1})$ (see $\partial[\iota_{n+1}]$ in Problem 2.2 for an example) and that of $\mathbb{Z}/m\mathbb{Z}$. Thus, an isomorphism between $\mathbb{Z}/m\mathbb{Z}$'s is not necessarily induced from $(f \times \mathbb{1})_*$ on H_{2n+1} in a canonical recipe. Indeed, if the splitting was natural, then any isomorphism induced by $f \times \mathbb{1}$ would map 0 to $H_{n+1}(S^{n+1}) \otimes H_n(M)$ and map $\operatorname{Tor}_1^{\mathbb{Z}}(H_n(M), H_n(M))$ to 0 simultaneously, which turns out to be impossible.

To conclude, the splitting in Künneth formula is not natural with respect to the change of any topological space in the product. □

Homework 7

Problem 7.1 (Hatcher, §3.1, Problem 11)**.** Let X be a Moore space $M(\mathbb{Z}/m\mathbb{Z}, n)$ obtained from S^n by attaching a cell e^{n+1} by a map of degree m.

(1) Show that the quotient map $X \to X/S^n = S^{n+1}$ induces the trivial map on $\tilde{H}_i(\textbf{-};\mathbb{Z})$ for all i, but not on $H^{n+1}(\textbf{-},\mathbb{Z})$. Deduce that the splitting in the universal coefficient theorem for cohomology cannot be natural.

(2) Show that the inclusion $S^n \hookrightarrow X$ induces the trivial map on $\tilde{H}^i(\textbf{-};\mathbb{Z})$ for all i, but not on $H_n(\textbf{-};\mathbb{Z})$.

Solution. (1) By construction, the cellular structure of X indicates that $\tilde{H}_i(X,\mathbb{Z})$ is only supported at degree n with $\tilde{H}_n(X,\mathbb{Z}) \cong \mathbb{Z}/m\mathbb{Z}$ (c.f. Problem 6.5). As for $X/S^n = S^{n+1}$, the reduced homology $\tilde{H}_i(S^{n+1},\mathbb{Z})$ is only supported at degree 0 and $n+1$ with $\tilde{H}_0(S^{n+1},\mathbb{Z}) \cong \tilde{H}_{n+1}(S^{n+1},\mathbb{Z}) \cong \mathbb{Z}$. Thus, the homological map induced by $q\colon X \to X/S^n$ must be trivial. As for $H^{n+1}(\textbf{-},\mathbb{Z})$, using the universal coefficient theorem for cohomology, we have

$$\begin{aligned} H^{n+1}(X;\mathbb{Z}) &\cong \operatorname{Ext}^1_{\mathbb{Z}}(H_n(X),\mathbb{Z}) \oplus \operatorname{Hom}(H_{n+1}(X),\mathbb{Z}) \\ &\cong \mathbb{Z}/m\mathbb{Z} \oplus 0 = \mathbb{Z}/m\mathbb{Z}, \end{aligned}$$

as well as

$$\begin{aligned} H^{n+1}(S^{n+1};\mathbb{Z}) &\cong \operatorname{Ext}^1_{\mathbb{Z}}(H_n(S^{n+1}),\mathbb{Z}) \oplus \operatorname{Hom}(H_{n+1}(S^{n+1}),\mathbb{Z}) \\ &\cong 0 \oplus \mathbb{Z} = \mathbb{Z}. \end{aligned}$$

Therefore, at a cohomological level, the induced map of $q\colon X \to X/S^n$ can be written as $q^*\colon \mathbb{Z} \to \mathbb{Z}/m\mathbb{Z}$. It suffices to check this is nonzero. But for this, q^* turns out to be surjective because q collapses all cells of dimension less than $n+1$ and preserves the $(n+1)$-cell.

(2) Similar to part (1), at the level of reduced cohomology, the inclusion $\iota\colon S^n \to X$ induces $\iota^*\colon \tilde{H}^i(X,\mathbb{Z}) \to \tilde{H}^i(S^n,\mathbb{Z})$, which could only be nontrivial at degree $i = n$. However, at degree n this is read as $\mathbb{Z}/m\mathbb{Z} \to \mathbb{Z}$, and hence must be trivial. As for $H_n(\textbf{-},\mathbb{Z})$, note that ι preserves the n-cells, and the n-cells for S^n and X are respectively the generators of $H_n(S^n,\mathbb{Z}) \cong \mathbb{Z}$ and $H_n(X,\mathbb{Z}) \cong \mathbb{Z}/m\mathbb{Z}$. It follows that $g_*\colon \mathbb{Z} \to m\mathbb{Z}$ is nontrivial. □

Problem 7.2 (Hatcher, §3.2, Problem 2)**.** Using the cup product

$$H^k(X,A;R) \times H^l(X,B;R) \longrightarrow H^{k+l}(X,A\cup B;R),$$

show that if X is the union of contractible open subsets A and B, then all cup products of positive-dimensional classes in $H^*(X;R)$ are zero. This applies in particular if X is a suspension. Generalize to the situation that X is the union of n contractible open subsets, to show that all n-fold cup products of positive-dimensional classes are zero.

Solution. Note that the given cup product fits in the following diagram which is always commutative:

$$\begin{array}{ccc} H^k(X,A;R) \times H^l(X,B;R) & \longrightarrow & H^{k+l}(X,A\cup B;R) \\ \downarrow\cong & & \downarrow \\ H^k(X;R) \times H^l(X;R) & \longrightarrow & H^{k+l}(X;R). \end{array}$$

The left vertical map is a priori not necessarily an isomorphism, but it becomes an isomorphism whenever both A, B are contractible. Indeed, since A, B are contractible, we have $\tilde{H}^i(A) = \tilde{H}^i(B) = 0$, and hence $H^i(X) \cong H^i(X,A) \cong H^i(X,B)$ for all i. As we only consider positive-dimensional classes, we may assume $k, l > 0$, and thus $k+l > 0$. For positive degrees, the assumption $X = A \cup B$ implies

$$H^{k+l}(X,A\cup B;R) \cong H^{k+l}(X,X;R) \cong H^{k+l}(X/X;R) = 0.$$

Therefore, the commutativity of the diagram forces the lower horizontal map $H^k(X;R)\times H^l(X;R) \to H^{k+l}(X;R)$ to be zero. This proves the first assertion. In general, we only need to upgrade the diagram above to an n-fold setting. If $X = A_1 \cup \cdots \cup A_n$ is a union of n contractible spaces, the same argument applies by noting that $H^*(X,A_j;R) \cong H^*(X;R)$ for each $1 \leqslant j \leqslant n$, so as to make the left vertical map an isomorphism again. □

Problem 7.3 (Hatcher, §3.2, Problem 6)**.** Use cup products to compute the map $H^*(\mathbb{CP}^n;\mathbb{Z}) \to H^*(\mathbb{CP}^n;\mathbb{Z})$ induced by the map $\mathbb{CP}^n \to \mathbb{CP}^n$ that is a quotient of the map $\mathbb{C}^{n+1} \to \mathbb{C}^{n+1}$, raising each coordinate to d-th power, $(z_0,\dots,z_n) \mapsto (z_0^d,\dots,z_n^d)$, for a fixed integer $d>0$.

Solution. Recall that we always have

$$H^*(\mathbb{CP}^n;\mathbb{Z}) \cong \mathbb{Z}[\alpha]/(\alpha^{n+1})$$

for $|a|=2$. We first consider the case $n=1$, where the map is $f_d\colon \mathbb{CP}^1 \to \mathbb{CP}^1,\ z_0 \mapsto z_0^d$. This map is defined by a complex polynomial of degree d, so it clearly has degree d as well (by applying the argument for Problem 4.1). So the induced cohomological map can be given by

$$f_d^*\colon H^*(\mathbb{CP}^1;\mathbb{Z}) \longrightarrow H^*(\mathbb{CP}^1;\mathbb{Z}), \quad \alpha \longmapsto d\alpha$$

in terms of $\mathbb{Z}[\alpha]/(\alpha^{n+1})$. (Here f_d^* maps $\mathbb{Z}$ to $\mathbb{Z}$ identically.)

Next, we upgrade this to $\mathbb{CP}^n$; by a property of the cup product, we always have $f_d^*(\alpha\cup\alpha) = f_d^*(\alpha)\cup f_d^*(\alpha)$, and hence

$$f_d^*\colon H^*(\mathbb{CP}^n;\mathbb{Z}) \longrightarrow H^*(\mathbb{CP}^n;\mathbb{Z}), \quad \alpha^n \longmapsto d^n\alpha^n.$$

Changing the notation, the pullback of each cohomological class $[\gamma]$ is given by $d[\gamma]$. □

Problem 7.4 (Hatcher, §3.2, Problem 8)**.** Let X be $\mathbb{CP}^2$ with a cell e^3 attached by a map $S^2 \to \mathbb{CP}^1 \hookrightarrow \mathbb{CP}^2$ of degree p, and let $Y = M(\mathbb{Z}/p\mathbb{Z},2)\vee S^4$. Thus X and Y have the same 3-skeleton but differ in the way their 4-cells are attached. Show that X and Y have isomorphic cohomology rings with $\mathbb{Z}$-coefficients but not with $\mathbb{Z}/p\mathbb{Z}$-coefficients.

Solution. We split the solution into the following two parts.

Step I (Cohomology groups)**.** We first compute the cohomology groups of X and Y with $\mathbb{Z}$-coefficients. Recall that the CW structure of $\mathbb{CP}^2$ consists of a 0-cell e^0, a 2-cell e^2, together with a 4-cell e^4. After attaching a 3-cell e^3 with degree p to it, the cellular chain complex of X is given by

$$0 \longrightarrow \Lambda \xrightarrow{0} \Lambda \xrightarrow{p} \Lambda \longrightarrow 0 \longrightarrow \Lambda \longrightarrow 0.$$

Here all boundary maps are zero except for $d_3 = p$. On the other hand, by construction, Y is the Moore space attached with e^4 of degree 1, so the cellular chain complex of Y is the same as above. Reversing the chain complex, the cellular cochain complexes for both X and Y are identical and written as

$$0 \longleftarrow \Lambda \xleftarrow{0} \Lambda \xleftarrow{p} \Lambda \longleftarrow 0 \longleftarrow \Lambda \longleftarrow 0.$$

Here, the first Λ from the left sits in degree 4. Taking the cohomology, we deduce

$$H^i(X;\Lambda) \cong H^i(Y;\Lambda) \cong \begin{cases} \Lambda, & i=4,\\ \Lambda/p\Lambda, & i=3,\\ \Lambda[p], & i=2,\\ 0, & i=1,\\ \Lambda, & i=0.\end{cases}$$

This holds for both $\Lambda=\mathbb{Z}$ or $\mathbb{Z}/p\mathbb{Z}$. Here $\Lambda[p]$ is the submodule of Λ annihilated by p, which is 0 for $\Lambda=\mathbb{Z}$ and is $\mathbb{Z}/p\mathbb{Z}$ for $\Lambda=\mathbb{Z}/p\mathbb{Z}$.

Step II (Cohomology rings)**.** For cohomology rings, we claim that $H^*(X;\mathbb{Z})$ and $H^*(Y;\mathbb{Z})$ are isomorphic, whereas $H^*(X;\mathbb{Z}/p\mathbb{Z})$ and $H^*(Y;\mathbb{Z}/p\mathbb{Z})$ are differed by the structure of cup product. For $\Lambda=\mathbb{Z}$ the higher cohomologies (ignoring H^0) of X and Y are concentrated in degree $[3,4]$, which forces all cup products to be trivial. It follows that

$$H^*(X;\mathbb{Z}) \cong H^*(Y;\mathbb{Z}).$$

Now we construct a nontrivial cup product in case of $\mathbb{Z}/p\mathbb{Z}$-coefficients. By the previous step, we have $H^i(X;\mathbb{Z}/p\mathbb{Z}) \cong H^i(Y;\mathbb{Z}/p\mathbb{Z}) \cong \mathbb{Z}/p\mathbb{Z}$ for $i=2,3,4$. So we can consider for $- \in \{X,Y\}$ the pairing

$$H^2(-;\mathbb{Z}/p\mathbb{Z}) \times H^2(-;\mathbb{Z}/p\mathbb{Z}) \xrightarrow{\cup} H^4(-;\mathbb{Z}/p\mathbb{Z}).$$

For $Y = M(\mathbb{Z}/p\mathbb{Z}, 2) \vee S^4$, we have $H^i(Y; \mathbb{Z}/p\mathbb{Z}) \cong H^i(M(\mathbb{Z}/p\mathbb{Z}, 2); \mathbb{Z}/p\mathbb{Z}) \oplus H^i(S^4; \mathbb{Z}/p\mathbb{Z})$, so the pairing above consists of two parts, respectively given by cup product on $M(\mathbb{Z}/p\mathbb{Z}, 2)$ and that on S^4. However, we have $H^4(M(\mathbb{Z}/p\mathbb{Z}, 2); \mathbb{Z}/p\mathbb{Z}) = 0$ and $H^2(S^4; \mathbb{Z}/p\mathbb{Z}) = 0$, which means the two components of the cup product has zero source and zero target, respectively. Thus, we have shown that the cup product for Y is trivial when $\Lambda = \mathbb{Z}/p\mathbb{Z}$.

In contrast, the cup product pairing above for X is nontrivial when $\Lambda = \mathbb{Z}/p\mathbb{Z}$, because $H^*(\mathbb{CP}^2; \Lambda) \cong \Lambda[u]/(u^3)$ (with variable u of degree 2) is equipped with a nontrivial cup product structure $u^i \cup u^j = u^{i+j}$.

To conclude, X and Y have isomorphic cohomology rings with $\mathbb{Z}$-coefficients but not with $\mathbb{Z}/p\mathbb{Z}$-coefficients. □

Problem 7.5 (Hatcher, §3.2, Problem 11)**.** Using cup products, show that every map $S^{k+l} \to S^k \times S^l$ induces the trivial homomorphism $H_{k+l}(S^{k+l}) \to H_{k+l}(S^k \times S^l)$, assuming $k > 0$ and $l > 0$.

Solution. Dually, it suffices to show that

$$\mathrm{Hom}(H_{k+l}(S^k \times S^l), \mathbb{Z}) \longrightarrow \mathrm{Hom}(H_{k+l}(S^{k+l}), \mathbb{Z})$$

induced from $S^{k+l} \to S^k \times S^l$ is trivial. We claim that this map is further identified with

$$H^{k+l}(S^k \times S^l) \longrightarrow H^{k+l}(S^{k+l}).$$

This claim in fact a consequence of Poincaré duality. To rather verify this in an elementary way, one can apply the universal coefficient theorem to deduce that

$$H^{k+l}(-) \cong \mathrm{Ext}^1_{\mathbb{Z}}(H_{k+l+1}(-), \mathbb{Z}) \oplus \mathrm{Hom}(H_{k+l}(-), \mathbb{Z}),$$

where $-$ can be S^{k+l} or $S^k \times S^l$. But for $n \geqslant 1$, since $H^i(S^n)$ is either zero or $\mathbb{Z}$, its extension group vanishes; consequently, the extension group also vanishes for $S^k \times S^l$ by Künneth formula. This proves the claim above. Therefore, for our purpose, we only need to show that $H^{k+l}(S^k \times S^l) \to H^{k+l}(S^{k+l})$ is zero.

Note that every $S^{k+l} \to S^k \times S^l$ induces a commutative diagram of cup products

$$\begin{array}{ccc} H^i(S^k \times S^l) \times H^j(S^k \times S^l) & \xrightarrow{\quad \cup \quad} & H^{i+j}(S^k \times S^l) \\ \downarrow \qquad\qquad \downarrow & & \downarrow \\ H^i(S^{k+l}) \times H^j(S^{k+l}) & \xrightarrow{\quad \cup \quad} & H^{i+j}(S^{k+l}) \end{array}$$

where all the left vertical maps are induced cohomological maps. We need to show the right vertical map is zero when $i = k$ and $j = l$. But in this case, $H^k(S^{k+l}) = H^l(S^{k+l}) = 0$, and the desired triviality follows from the commutativity. □

Problem 7.6 (Hatcher, §3.2, Problem 12)**.** Show that the spaces $(S^1 \times \mathbb{CP}^\infty)/(S^1 \times \{x_0\})$ and $S^3 \times \mathbb{CP}^\infty$ have isomorphic cohomology rings with $\mathbb{Z}$ or any other coefficients.

Solution. We work over the coefficient ring Λ. Recall that $H^*(\mathbb{CP}^\infty; \Lambda) \cong \Lambda[u]$ with u of degree 2, and $H^*(S^n; \Lambda) \cong \Lambda[x_n]/(x_n^2)$ with variable x_n of degree n. Applying Künneth formula, we obtain

$$H^*(S^n \times \mathbb{CP}^\infty) \cong H^*(S^n) \otimes H^*(\mathbb{CP}^\infty) \cong \Lambda[x_n]/(x_n^2) \otimes \Lambda[u].$$

Now the ring $H^*(S^3 \times \mathbb{CP}^\infty)$ can be directly computed through the formula above. Notice that in $H^*(S^3 \times \mathbb{CP}^\infty)$, for $k \in \mathbb{Z}_{\geqslant 1}$, each class of degree $2k$ is generated by the self-cup-product of u for k times, and each class of degree $2k+1$ is generated by the cup product of a class of degree $2(k-1)$ together with x_3; there is no class of degree 1 in $H^*(S^3 \times \mathbb{CP}^\infty)$.

As for the cohomology ring of $(S^1 \times \mathbb{CP}^\infty)/(S^1 \times \{x_0\})$, notice that after quotient by $S^1 \times \{x_0\}$, the class x_1 of degree 1 vanishes whereas the cup product $x_1 \cup u$ of degree 3 survives. This construction gives rise to an isomorphism

$$H^*(S^3 \times \mathbb{CP}^\infty) \xrightarrow{\sim} H^*((S^1 \times \mathbb{CP}^\infty)/(S^1 \times \{x_0\}))$$

that maps x_3 to $x_1 \cup u$ and u to u, and vice versa.

Caution that the two spaces in this problem are in fact not homotopy equivalent (see [Hat02, §4.L, Problem 4]). □

Problem 7.7. Use definition of cup product on simplicial cohomology to compute the cohomology ring $H^*(\Sigma_g; \mathbb{Z})$ for a closed, orientable surface of genus g.

Solution. We know that the cellular structure of Σ_g consists of a 0-cell, a 2-cell, together with $2g$ 1-cells which are loops corresponding to the g "handles" on Σ_g, denoted by $a_1, b_1, \ldots, a_g, b_g$. Notice that the unique 2-cell is attached to 1-skeleton in a way that imposes the single relation $\prod_{i=1}^{g}[a_i, b_i] = 1$ in $\pi_1(\Sigma_g)$. Thus, the cellular cochain complex computes the cohomology as

$$H^i(\Sigma_g) \cong \begin{cases} \mathbb{Z}, & i = 2, \\ \mathbb{Z}^{\oplus 2g}, & i = 1, \\ \mathbb{Z}, & i = 0. \end{cases}$$

We choose cohomology generators as follows. For $1 \leqslant i \leqslant g$, let a_i^* and b_i^* be the classes of degree 1 dual to the loop a_i and b_i, respectively. Choose z to be an orientation class of degree 2 that generates $H^2(\Sigma_g)$, such that $z = a_i^* \cup b_i^*$ for all $1 \leqslant i \leqslant g$. Notice that the intersections between a_i's and b_j's exactly happen when $i = j$. So we have

$$a_i^* \cup a_j^* = b_i^* \cup b_j^* = a_i^* \cup b_j^* = 0$$

for $i \neq j$. To conclude,

$$H^*(\Sigma_g; \mathbb{Z}) \cong \frac{\mathbb{Z}\langle a_1^*, b_1^*, \ldots, a_g^*, b_g^* \rangle}{(a_i^* \cup b_i^* - a_j^* \cup b_j^*,\ a_i^* \cup a_j^*,\ b_i^* \cup b_j^*,\ a_i^* \cup b_j^*)_{i \neq j}}$$

gives the desired cohomology ring. □

Homework 8

Problem 8.1 (Hatcher, §3.3, Problem 6)**.** Given two disjoint connected n-manifolds M_1 and M_2, a connected n-manifold $M_1 \# M_2$, their connected sum, can be constructed by deleting the interiors of closed n-balls $B_1 \subset M_1$ and $B_2 \subset M_2$ and identifying the resulting boundary spheres ∂B_1 and ∂B_2 via some homeomorphism between them. (Assume that each B_i embeds nicely in a larger ball in M_i.)

(1) Show that if M_1 and M_2 are closed then there are isomorphisms
$$H_i(M_1 \# M_2; \mathbb{Z}) \cong H_i(M_1; \mathbb{Z}) \oplus H_i(M_2; \mathbb{Z})$$
for $0 < i < n$, with only one exception as follows. If both M_1 and M_2 are non-orientable, then $H_{n-1}(M_1 \# M_2; \mathbb{Z})$ is obtained from $H_{n-1}(M_1; \mathbb{Z}) \oplus H_{n-1}(M_2; \mathbb{Z})$ by replacing one of the two $\mathbb{Z}/2\mathbb{Z}$ summands by a $\mathbb{Z}$ summand.

(2) Show that
$$\chi(M_1 \# M_2) = \chi(M_1) + \chi(M_2) - \chi(S^n)$$
if M_1 and M_2 are closed.

Solution. (1) In the following, write $H_i(\mathbf{-}) = H_i(\mathbf{-}; \mathbb{Z})$ for simplicity. For $j = 1, 2$ denote $N_j := M_j \setminus \operatorname{Int}(B_j)$, that is, removing the interior of B_j from M_j. Then N_j is closed. Also, we have $N_1 \cap N_2 = S^{n-1}$ and $M_1 \# M_2 = N_1 \cup N_2$. Applying the Mayer–Vietoris long exact sequence to N_1, N_2, we obtain
$$\cdots \longrightarrow H_i(S^{n-1}) \longrightarrow H_i(N_1) \oplus H_i(N_2) \longrightarrow H_i(M_1 \# M_2) \longrightarrow H_{i-1}(S^{n-1}) \longrightarrow \cdots.$$
On the other hand, there is another long exact sequence
$$\cdots \longrightarrow H_i(S^{n-1}) \longrightarrow H_i(N_j) \longrightarrow H_i(N_j, S^{n-1}) \longrightarrow H_{i-1}(S^{n-1}) \longrightarrow \cdots.$$
Importantly, notice that $H_i(S^{n-1}) \to H_i(N_j)$ in the second sequence is the same as each component of $H_i(S^{n-1}) \to H_i(N_1) \oplus H_i(N_2)$ in the first sequence; indeed, it is induced by the natural embedding that identifies S^{n-1} with $\partial B_j \subset N_j$.

Recall that $H_i(S^{n-1}) = 0$ for $i \notin \{0, n-1\}$, in which case $H_i(N_j) \cong H_i(N_j, S^{n-1})$ from the second long exact sequence above. But $H_i(N_j, S^{n-1}) \cong H_i(N_j/S^{n-1}) \cong H_i(M_j)$, so we have $H_i(N_j) \cong H_i(M_j)$ for $0 < i < n-1$. Thus, when $0 < i < n-1$,
$$H_i(M_1 \# M_2) \cong H_i(N_1) \oplus H_i(N_2) \cong H_i(M_1) \oplus H_i(M_2).$$

Now it remains to consider $i = n-1$. In the ordinary case, at least one of M_1, M_2, say M_1 without loss of generality, is orientable. Then using the first sequence above, $H_{n-2}(S^{n-1}) = 0$ implies that $H_{n-1}(M_1 \# M_2)$ is the cokernel of $H_{n-1}(S^{n-1}) \to H_{n-1}(N_1) \oplus H_{n-1}(N_2)$. But as M_1 is orientable, we have $H_n(M_1) \cong H_{n-1}(S^{n-1}) \cong \mathbb{Z}$ and $H_{n-1}(N_1)$ is torsion-free; since N_1 is constructed by removing one n-cell of top dimension from M_1, we see $H_n(N_1) = 0$. It follows that, in the long exact sequence, the connection map $H_n(M_1) \to H_{n-1}(S^{n-1})$ is an isomorphism and $H_{n-1}(S^{n-1}) \to H_{n-1}(N_1)$ is zero. From this argument,
$$\begin{aligned} H_{n-1}(M_1 \# M_2) &\cong \operatorname{coker}(H_{n-1}(S^{n-1}) \to H_{n-1}(N_1) \oplus H_{n-1}(N_2)) \\ &\cong H_{n-1}(N_1) \oplus \operatorname{coker}(H_{n-1}(S^{n-1}) \to H_{n-1}(N_2)) \\ &\cong H_{n-1}(N_1) \oplus H_{n-1}(M_2). \end{aligned}$$
Again, as $H_{n-1}(S^{n-1}) \to H_{n-1}(N_1)$ is zero, we have $H_{n-1}(N_1) \cong H_{n-1}(M_1)$, which proves the desired isomorphism $H_{n-1}(M_1 \# M_2) \cong H_{n-1}(M_1) \oplus H_{n-1}(M_2)$.

We then consider the exceptional case. Suppose both M_1, M_2 are non-orientable. Recall that if a space X of dimension n is non-orientable, then its top homology $H_n(X) = 0$, and the torsion part of $H_{n-1}(X)$ is exactly $\mathbb{Z}/2\mathbb{Z}$. We modify the second long exact sequence at the beginning by replacing N_j with $N_1 \cup N_2 = M_1 \# M_2$ to attain the following:
$$\cdots \longrightarrow H_i(S^{n-1}) \longrightarrow H_i(M_1 \# M_2) \longrightarrow H_i(M_1 \# M_2, S^{n-1}) \longrightarrow H_{i-1}(S^{n-1}) \longrightarrow \cdots$$
in which we have
$$H_i(M_1 \# M_2, S^{n-1}) \cong H_i(M_1 \# M_2/S^{n-1}) \cong H_i(M_1 \vee M_2) \cong H_i(M_1) \oplus H_i(M_2).$$

In the exceptional case, the vanishings $H_n(M_j) = 0$ and $H_{n-2}(S^{n-1}) = 0$ truncate the new long exact sequence into a short one at degree $n-1$, read as

$$0 \longrightarrow H_{n-1}(S^{n-1}) \longrightarrow H_{n-1}(M_1 \# M_2) \longrightarrow H_{n-1}(M_1) \oplus H_{n-1}(M_2) \longrightarrow 0.$$

Since the torsion part of $H_{n-1}(M_j)$ is exactly $\mathbb{Z}/2\mathbb{Z}$, we write

$$H_{n-1}(M_j) \cong \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}^{\oplus r_j}$$

for some $r_j \in \mathbb{Z}_{\geqslant 0}$. Now we conclude $H_{n-1}(M_1) \oplus H_{n-1}(M_2)$ is a quotient of $H_{n-1}(M_1 \# M_2)$, and their ranks over $\mathbb{Z}$ are differed by 1 because of $H_{n-1}(S^{n-1}) \cong \mathbb{Z}$. Therefore, we can get $H_{n-1}(M_1 \# M_2)$ by replacing one of the two $\mathbb{Z}/2\mathbb{Z}$'s inside $H_{n-1}(M_1) \oplus H_{n-1}(M_2)$ by a $\mathbb{Z}$.

(2) By Mayer–Vietoris sequence for N_1, N_2 in (1), we have

$$\chi(M_1 \# M_2) = \chi(N_1) + \chi(N_2) - \chi(S^{n-1}).$$

Note that N_j is obtained from M_j by removing an n-cell B_j of top dimension, so the rank of $H_n(N_j)$ equals that of $H_n(M_j)$ decreased by 1, and hence

$$\chi(N_j) = \chi(M_j) - (-1)^n.$$

On the other hand, using the cohomology of S^k, we immediately get $\chi(S^k) = 1 + (-1)^k$. Combining the ingredients above, we see

$$\begin{aligned}\chi(M_1 \# M_2) &= \chi(M_1) + \chi(M_2) - 2 \cdot (-1)^n - (1 + (-1)^{n-1}) \\ &= \chi(M_1) + \chi(M_2) + (-1 - (-1)^n) \\ &= \chi(M_1) + \chi(M_2) - \chi(S^n).\end{aligned}$$

This only depends on the condition that M_1, M_2 are both closed, and is independent of the orientability. □

Problem 8.2 (Hatcher, §3.3, Problem 8)**.** For a map $f\colon M \to N$ between connected closed orientable n-manifolds, suppose there is a ball $B \subset N$ such that $f^{-1}(B)$ is the disjoint union of balls B_i each mapped homeomorphically by f onto B. Show the degree of f is $\sum_i \varepsilon_i$ where ε_i is $+1$ or -1 according to whether $f\colon B_i \to B$ preserves or reverses local orientations induced from given fundamental classes $[M]$ and $[N]$.

Solution. By definition we have $f_*([M]) = \deg f \cdot [N]$ for $f_*\colon \tilde{H}_n(M) \to \tilde{H}_n(N)$, and $\deg f$ can be computed through local restriction over $B \subset N$, i.e. it suffices to compute $f_*(f^{-1}([B]))$, where $[B] \coloneqq [N]|_B$. More precisely, by excision f_* restricts to the relative homology $H_n(B_i, \partial B_i) \to H_n(B, \partial B)$ on each B_i, which is isomorphic to $\mathbb{Z} \to \mathbb{Z}$. But by construction, $f^{-1}(B) = \bigsqcup_{i \in I} B_i$ and f is a homeomorphism on each B_i. Thus, $f_*\colon \mathbb{Z} \cong H_n(B_i, \partial B_i) \to H_n(B, \partial B) \cong \mathbb{Z}$ is an isomorphism, which must be given by the multiplication by $\varepsilon_i \in \{\pm 1\}$. To sum up, we have

$$f_*(f^{-1}([B])) = \sum_{i \in I} f_*([M]|_{B_i}) = \sum_{i \in I} \varepsilon_i [B] = \deg f \cdot [B],$$

which implies the desired relation $\deg f = \sum_{i \in I} \varepsilon_i$. □

Problem 8.3 (Hatcher, §3.3, Problem 11)**.** If M_g denotes the closed orientable surface of genus g, show that degree 1 maps $M_g \to M_h$ exist if and only if $g \geqslant h$.

Solution. Suppose $g \geqslant h$. Then there is a decomposition $M_g \approx M_h \# M_{g-h}$ (see Problem 8.1 for the connected sum). So there is a degree 1 map $M_g \to M_h$ collapsing M_{g-h} (containing the boundary S^1 of the connected sum) into a point and being homeomorphic on M_h.

Conversely, assume $M_g \to M_h$ of degree 1 exists. It then induces a homological isomorphism $f_*\colon H_2(M_g) \to H_2(M_h)$ sending $[M_g]$ to $[M_h]$. Take any nonzero $\alpha \in H^1(M_h)$. Then by Poincaré duality we compute the cap product $f^*(\alpha) \cap [M_g] = \alpha \cap f_*([M_g]) = \alpha \cap [M_h]$, which is nonzero as well. In particular, we have $f^*(\alpha) \neq 0$, showing that $f^*\colon H^1(M_h) \to H^1(M_g)$ is an injection. On the other hand, at degree 1 we have $H^1(M_h) \cong \mathbb{Z}^{2h}$ and $H^1(M_g) \cong \mathbb{Z}^{2g}$, so we have an injective map $\mathbb{Z}^{2h} \hookrightarrow \mathbb{Z}^{2g}$. It follows that $g \geqslant h$. □

Appendix A. Extra problems

Problem A.1 (Hatcher, §3.3, Problem 25)**.** Show that if a closed orientable manifold M of dimension $2k$ has $H_{k-1}(M;\mathbb{Z})$ torsion-free, then $H_k(M;\mathbb{Z})$ is also torsion-free.

Problem A.2 (Hatcher, §3.3, Problem 26)**.** Compute the cup-product structure in $H^*(S^2\times S^8 \# S^4\times S^6;\mathbb{Z})$, and in particular show that the only non-trivial cup products are those dictated by Poincaré duality. [See Exercise 6. The result has an evident generalization to connected sums of $S^i\times S^{n-i}$'s for fixed n and varying i.]

Problem A.3 (Hatcher, §3.3, Problem 27)**.** Show that after a suitable change of basis, a skew-symmetric nonsingular bilinear form over $\mathbb{Z}$ can be represented by a matrix consisting of 2×2 blocks $\left(\begin{smallmatrix}0 & -1\\ 1 & 0\end{smallmatrix}\right)$ along the diagonal and zeros elsewhere. [For the matrix of a bilinear form, the following operation can be realized by a change of basis: Add an integer multiple of the i-th row to the j-th row and add the same integer multiple of the i-th column to the j-th column. Use this to fix up each column in turn. Note that a skew-symmetric matrix must have zeros on the diagonal.]

Problem A.4 (Hatcher, §3.3, Problem 28)**.** Show that a nonsingular symmetric or skew-symmetric bilinear pairing over a field F, of the form $F^n\times F^n\to F$, cannot be identically zero when restricted to all pairs of vectors $\mathbf{v},\mathbf{w}$ in a k-dimensional subspace $V\subset F^n$ if $k>n/2$.

Solution. Note that all vector spaces are finite dimensional. Let $B\colon F^n\times F^n\to F$ be the given nondegenerate bilinear symmetric or skew-symmetric form. □

Problem A.5 (Hatcher, §3.3, Problem 29)**.** Use the preceding problem to show that if the closed orientable surface M_g of genus g retracts onto a graph $X\subset M_g$, then $H_1(X)$ has rank at most g. Deduce an alternative proof of Exercise 13 from this, and construct a retraction of M_g onto a wedge sum of k circles for each $k\leqslant g$.

Problem A.6 (Hatcher, §3.3, Problem 32)**.** Show that a compact manifold does not retract onto its boundary.

Problem A.7 (Hatcher, §3.3, Problem 33)**.** Show that if M is a compact contractible n-manifold then ∂M is a homology $(n-1)$-sphere, that is, $H_i(\partial M;\mathbb{Z})\cong H_i(S^{n-1};\mathbb{Z})$ for all i.

Problem A.8 (MIT 18.906, Problem Set I, Problem 1)**.**

(1) Show that any limit can be expressed as an equalizer of two maps between products.
(2) Let $\mathcal{C}$ and $\mathcal{D}$ be two categories and $F\colon\mathcal{C}\to\mathcal{D}$ and $G\colon\mathcal{D}\to\mathcal{C}$ two functors. In class we said that an adjunction between F and G is an isomorphism

$$\mathcal{D}(FX,Y)\cong\mathcal{C}(X,GY)$$

that is natural in both variables. Show that this is equivalent to giving natural transformations

$$\alpha_X\colon X\to GFX,\quad \beta_Y\colon FGY\to Y,$$

such that

$$\beta_{FX}\circ F\alpha_X=\mathbb{1}_{FX},\quad G\beta_Y\circ\alpha_{GY}=\mathbb{1}_{GY}.$$

(3) Suppose that F and F' are both left adjoint to $G\colon\mathcal{D}\to\mathcal{C}$. Show that there is a unique natural isomorphism $F\to F'$ that is compatible with the adjunction.

Problem A.9 (MIT 18.906, Problem Set I, Problem 2)**.**

(1) We have used the notation Z^X for a mapping object in a Cartesian closed category, and $\mathcal{C}^{\mathcal{I}}$ for the category of functors to $\mathcal{C}$ from a small category $\mathcal{I}$. Does this constitute a conflict of notation? Explain.
(2) Let $\mathcal{C}$ be a Cartesian closed category.
    (i) Verify the exponential laws: construct natural isomorphisms

$$Z^{X\times Y}\cong(Z^X)^Y,\quad (Y\times Z)^X\cong Y^X\times Z^X.$$

The first of these shows that the adjunction bijection

$$\mathcal{C}(X\times Y,Z)\cong\mathcal{C}(Y,Z^X)$$

"enriches" to an isomorphism in $\mathcal{C}$. The second says that the product in $\mathcal{C}$ is actually an "enriched" product.

(ii) Construct a "composition" natural transformation

$$Y^X \times Z^Y \to Z^X$$

using the evaluation maps, and show that it is associative and unital.

(3) Construct left and right adjoints to the forgetful functor

$$u\colon \mathsf{Top} \longrightarrow \mathsf{Set}.$$

and conclude that for any small category I, the limit and the colimit of a functor $X\colon I \to \mathsf{Top}$ consists of the corresponding limit or colimit of underlying sets endowed with a suitable topology.

(4) Show that the colimit (in Top) of any diagram of k-spaces is again a k-space, and serves as the colimit in $k\mathsf{Top}$. (Suggestion: Show that in Top any coproduct of k-spaces is a k-space and that any quotient of a k-space is a k-space, and then use the dual of **1(a)**.)

Problem A.10. Consider the functor

$$H^k(-;\mathbb{Z})\colon \mathsf{HoTop} \to \mathsf{Set}.$$

Show that this functor is not corepresentable if $k > 0$. (How does a corepresentable functor behave under product?)

Problem A.11.

(1) Consider the functor

$$- \times X\colon \mathsf{Set} \to \mathsf{Set}.$$

Show that this functor has no left adjoint if X has more than two elements. (Hint: right adjoints preserve limits.)

(2) Show that the category $\mathsf{Set}^{\mathrm{op}}$ is not Cartesian closed.

(3) Show that the category of Abelian groups is not Cartesian closed.

Problem A.12 (MIT 18.906, Problem Set I, Problem 4)**.** Show that the fiber bundle $\mathrm{SO}(n) \to S^{n-1}$ sending an orthogonal matrix with determinant 1 to its first column has a section if and only if S^{n-1} is parallelizable. What are the situations for $n = 3$ and $n = 4$ respectively?

Problem A.13 (MIT 18.906, Problem Set II, Problem 6)**.**

(1) Show that weak equivalences satisfy "2 out of 3": in

$$X \xrightarrow{\ f\ } Y \xrightarrow{\ g\ } Z, \qquad gf\colon X \to Z$$

if two of f, g, and gf are weak equivalences then so is the third.

(2) Let $f\colon \Sigma X \to Y$ be a pointed map, and let $\hat{f}\colon X \to \Omega Y$ be its adjoint. Construct a map $g\colon \mathrm{C}X \to \mathrm{P}Y$ from the cone to the path space $\mathrm{P}Y = Y_*^I$ such that the diagram below commutes.

$$\begin{array}{ccccc} X & \longrightarrow & \mathrm{C}X & \longrightarrow & \Sigma X \\ {\scriptstyle \hat{f}}\downarrow & & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle f} \\ \Omega Y & \longrightarrow & \mathrm{P}Y & \longrightarrow & Y \end{array}$$

Problem A.14. Solve the following problems.

(1) Use definition to show that the Hopf map

$$\eta\colon S^3 \longrightarrow S^2, \quad (z_1, z_2) \longmapsto z_1/z_2$$

is a fibration. Here we treat S^3 as the unit sphere in $\mathbb{C}^2$ and treat S^2 as the one-point compactification of $\mathbb{C}$.

(2) Recall that a point in $\mathbb{CP}^n$ corresponds to a line in $L \subset \mathbb{C}^{n+1}$. Consider the space $E \coloneqq \{(L, v) \in \mathbb{CP}^n \times \mathbb{C}^{n+1} \colon v \in L\}$ and the map

$$p \colon E \longrightarrow \mathbb{CP}^n, \quad (L, v) \longmapsto L.$$

Use definition to show that p is a fiber bundle.

Problem A.15. Given two maps $f, g \colon X \to Y$. Suppose f is homotopic to g. Show that the homotopy fiber of f is homotopy equivalent to the homotopy fiber of g.

Problem A.16. Let X be a CW complex and let $p \colon \widetilde{X} \to X$ be a covering map. Pick base points $x \in X$ and $\tilde{x} \in p^{-1}(x)$.

(1) Show that the induced map $p_* \colon \pi_n(\widetilde{X}, \tilde{x}) \to \pi_n(X, x)$ is an isomorphism for any $n \geqslant 2$.
(2) Suppose p is a universal covering map. Then we can identify $\pi_1(X, x)$ with the group of deck transformations of p. Given $g \in \pi_1(X, x)$, let $\tilde{g} \colon \widetilde{X} \to \widetilde{X}$ be the deck transformation corresponding to g. Pick any path γ in $\widetilde{X}$ from $\tilde{x}$ to $\tilde{g}(\tilde{x})$. Consider the following composition

$$\begin{aligned} \pi_n(X, x) \xrightarrow{p_*^{-1}} \pi_n(\widetilde{X}, \tilde{x}) \xrightarrow{\tilde{g}_*} \pi_n(\widetilde{X}, \tilde{g}(\tilde{x})) \\ \xrightarrow{\gamma^\sharp} \pi_n(\widetilde{X}, \tilde{x}) \xrightarrow{p_*} \pi_n(X, x). \end{aligned}$$

Show that this is exactly the action of g on $\pi_n(X, x)$.
(3) Show that the space $\mathbb{RP}^n$ is simple if and only if n is odd. Here simple means that the fundamental group acts trivially on the higher homotopy groups.

Problem A.17. Using cohomology ring to show that the Hopf map $\eta \colon S^3 \to S^2$ is a nontorsion element in $\pi_3(S^2)$.

Appendix B. Final Exam

The exam candidates are to complete the 8 problems below with 3 hours.

Problem B.1. Consider the quotient space $X = (S^3 \times [0,1])/\sim$, where the equivalence relation $\sim$ is defined for any $v \in S^3$ by $(v,0) \sim (-v,0)$ and $(v,1) \sim (-v,1)$. Compute the homology groups of X with coefficient $\mathbb{Z}$.

Solution. Consider the decomposition $X = X_1 \cup X_2$ where $X_1 = (S^3 \times [0,1))/\sim$ and $X_1 = (S^3 \times (0,1])/\sim$. Then up to deformation retract, we have a homotopy equivalence $X_1 \approx (S^3 \times \{0\})/\sim$, so that $X_1 \approx \mathbb{RP}^3$. For the same reason we have $X_2 \approx \mathbb{RP}^3$. On the other hand, we obtain $X_1 \cap X_2 \approx S^3$. Then by the Mayer–Vietoris sequence, it follows that

$$\cdots \longrightarrow H_k(S^3) \xrightarrow{(i_{1,*}, i_{2,*})} H_k(\mathbb{RP}^3) \oplus H_k(\mathbb{RP}^3) \longrightarrow H_k(X) \longrightarrow H_{k-1}(S^3) \longrightarrow \cdots,$$

where $i_j \colon X_1 \cap X_2 \hookrightarrow X_j$ for $j \in \{1,2\}$.

Now it suffices to compute the map $(i_{1,*}, i_{2,*})$. Notice that along the deformation retract, both i_1 and i_2 are homotopic to the quotient map defined by $S^3 \to \mathbb{RP}^3$, and therefore

$$(i_{1,*}, i_{2,*}) \colon H_3(S^3) \longrightarrow H_3(\mathbb{RP}^3) \oplus H_3(\mathbb{RP}^3), \quad 1 \longmapsto (2,2).$$

To conclude, we have

$$H_k(X) \cong \begin{cases} \mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}, & i = 3, \\ 0, & i = 2, \\ \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}, & i = 1, \\ \mathbb{Z}, & i = 0. \end{cases}$$

For dimension reason, the $H_k(X)$ vanishes for $k > 3$. □

Problem B.2. Are the spaces $\mathbb{CP}^3 \# \mathbb{CP}^3$ homotopic to $S^2 \times \mathbb{CP}^2$? Answer the question with a justification.

Solution. The two spaces are not homotopic. Indeed, consider that

$$\mathbb{CP}^3 \# \mathbb{CP}^3 = (\mathbb{CP}^3 \backslash D^6) \cup_{S^5} (\mathbb{CP}^3 \backslash D^6).$$

Using the Mayer–Vietoris sequence, we obtain

$$\begin{aligned} H^k(\mathbb{CP}^3 \# \mathbb{CP}^3) &\cong H^k(\mathbb{CP}^3 \backslash D^6) \oplus H^k(\mathbb{CP}^3 \backslash D^6) \\ &\cong H^k(\mathbb{CP}^3) \oplus H^k(\mathbb{CP}^3) \end{aligned}$$

for all $k \leqslant 4$. The group isomorphisms above are induced by embeddings $\mathbb{CP}^3 \backslash D^6 \hookrightarrow \mathbb{CP}^3 \hookrightarrow \mathbb{CP}^3 \# \mathbb{CP}^3$, and hence are ring homomorphisms. As a consequence, the map

$$H^2(\mathbb{CP}^3 \# \mathbb{CP}^3) \longrightarrow H^3(\mathbb{CP}^3 \# \mathbb{CP}^3), \quad x \longmapsto x^2$$

is exactly given in terms of $\mathbb{Z} \oplus \mathbb{Z} \to \mathbb{Z} \oplus \mathbb{Z}$, $(a,b) \mapsto (a^2, b^2)$; in particular, this map is injective.

On the other hand, consider the projection $p \colon S^2 \times \mathbb{CP}^2 \to S^2$. Let y be a generator of $H^2(S^2)$. Applying the Künneth formula, we have $p^*(y) \neq 0$, whereas $(p^*(y))^2 = p^*(y^2) = 0$. It follows that the cohomology rings of the two spaces are not isomorphic, which completes the proof of our claim. □

Problem B.3. Show that any non-orientable closed surface cannot be embedded into $\mathbb{R}^3$.

Solution. Assume for the sake of contradiction that there exists a subset $K \subset \mathbb{R}^3$ homeomorphic to a non-orientable closed surface. Then by Alexander duality,

$$\tilde{H}^2(K;\mathbb{Z}) \cong \tilde{H}_0(S^3 \backslash K;\mathbb{Z}).$$

However, applying the universal coefficient theorem, we know that

$$\tilde{H}^2(K;\mathbb{Z}) \cong \operatorname{Ext}(H_1(K;\mathbb{Z}),\mathbb{Z}) \cong \operatorname{Tor}(H_1(K;\mathbb{Z})) \cong \mathbb{Z}/2\mathbb{Z}.$$

So we get $\tilde{H}_0(S^3 \backslash K;\mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z}$, which is a contradiction as any homology of degree 0 must be torsion-free. □

Problem B.4. Let M be an orientable closed manifold of dimension $4k+2$. Show that the Euler characteristic of M must be an even integer.

Solution. By Poincaré duality, we have

$$H^i(M;\mathbb{Q}) \cong H_{4k+2-i}(M;\mathbb{Q}).$$

In particular, for Betti numbers we have $b_i(M) = b_{4k+2-i}(M)$. It follows that the Euler characteristic $\chi(M) \equiv b_{2k+1}(M) \bmod 2$, so we only need to show that $b_{2k+1}(M)$ is even. For this, consider the pairing

$$\boldsymbol{I}\colon H^{2k+1}(M;\mathbb{Q}) \times H^{2k+1}(M;\mathbb{Q}) \longrightarrow \mathbb{Q}, \quad (a,b) \longmapsto \langle a \cup b, [M]\rangle.$$

This is a non-degenerate quadratic form by Poincaré duality. Due to the commutativity of cup product, we see $\boldsymbol{I}$ is anti-symmetric. Choosing a basis of $H^{2k+1}(M;\mathbb{Q})$, we see $\boldsymbol{I}$ is represented by an anti-symmetric invertible matrix I with

$$\det I = \det I^T = \det(-I) = (-1)^{b_{2k+1}(M)} \det I.$$

This proves $b_{2k+1}(M) \equiv 0 \bmod 2$ as desired. □

Problem B.5. Let X be a 3-dimensional CW complex with 2-dimension skeleton $S^2 \vee S^2$. Assume X has the unique 3-dimensional cell with gluing map

$$f\colon \partial D^3 = S^2 \longrightarrow S^2 \vee S^2.$$

Consider its induced map $f_*\colon H_2(S^2;\mathbb{Z}) \to H_2(S^2 \vee S^2;\mathbb{Z})$, which is isomorphic to $f_*\colon \mathbb{Z} \to \mathbb{Z}\oplus\mathbb{Z}$. Suppose $f_*(1) = (m,n)$ with $m,n \in \mathbb{Z}$. Show that X is homotopic to S^2 if and only if m,n are coprime integers.

Solution. Consider the cellular chain complex

$$0 \longrightarrow \mathbb{Z} \xrightarrow{d_3} \mathbb{Z}\oplus\mathbb{Z} \longrightarrow 0 \longrightarrow \mathbb{Z} \longrightarrow 0,$$

where $d_3\colon 1 \mapsto (m,n)$ by assumption. Then $H_2(X) \cong (\mathbb{Z}\oplus\mathbb{Z})/\gcd(m,n)$. Whenever $X \approx S^3$, we have the vanishing of $H_2(X)$; it then follows that $\gcd(m,n) = 1$.

Conversely, assume $\gcd(m,n) = 1$. Now for any k we have $H_k(X) \cong H_k(S^2)$. Notice that X is simply connected, so that the Hurewicz map

$$\pi_2(X) \xrightarrow{\sim} H_2(X), \quad [f] \longmapsto f_*([S^2])$$

is an isomorphism. Consequently, there exists $f\colon S^2 \to X$ inducing the isomorphisms on all homology groups. By Whitehead's theorem, this map gives a homotopy equivalence. □

Problem B.6. Compute $\pi_k(\mathbb{CP}^{1012})$ for all $k \leqslant 2025$.

Solution. Consider the fibre bundle

$$S^1 \hookrightarrow S^{2025} \longrightarrow \mathbb{CP}^{1012},$$

where S^1 acts on S^{2025} by complex multiplications by unit vectors; we view $\mathbb{CP}^{1012}$ as the quotient of S^{2025} by the free S^1-action. Then there is a long exact sequence

$$\cdots \longrightarrow \pi_k(S^1) \longrightarrow \pi_k(S^{2025}) \longrightarrow \pi_k(\mathbb{CP}^{1012}) \longrightarrow \pi_{k-1}(S^1) \longrightarrow \cdots.$$

On the other hand, recall that

$$\pi_k(S^1) \cong \begin{cases} \mathbb{Z}, & k=1, \\ 0, & k \geqslant 2, \end{cases} \qquad \pi_k(S^{2025}) \cong \begin{cases} \mathbb{Z}, & k = 2025, \\ 0, & 1 \leqslant k \geqslant 2024. \end{cases}$$

Combining the facts above with the long exact sequence, one can conclude that

$$\pi_k(\mathbb{CP}^{1012}) \cong \begin{cases} \mathbb{Z}, & k = 2 \text{ or } 2025, \\ 0, & k = 1 \text{ or } 3 \leqslant k \geqslant 2024. \end{cases}$$

□

Problem B.7. Determine the cardinality of the set $[T^2, \mathbb{RP}^3]$ consisting of homotopy equivalence classes of maps $T^2 \to \mathbb{RP}^3$.

Solution. Recall that for Eilenberg–McLane space $K(\mathbb{Z}/2\mathbb{Z},1) \approx \mathbb{RP}^\infty$, we have

$$[T^2, \mathbb{RP}^\infty] \cong H^1(T^2; \mathbb{Z}/2\mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}.$$

In particular, there are 4 elements in $[T^2, \mathbb{RP}^\infty]$. Now consider the embedding $i\colon \mathbb{RP}^3 \hookrightarrow \mathbb{RP}^\infty$ which induces

$$i_*\colon [T^2, \mathbb{RP}^3] \longrightarrow [T^2, \mathbb{RP}^\infty].$$

We claim that i_* is a bijection, so that the cardinality of $[T^2, \mathbb{RP}^3]$ is 4 as well.

To show the surjectivity of i_*, take any $f\colon T^2 \to \mathbb{RP}^\infty$. By CW approximation theorem, f is homotopic to a CW map $f'\colon T^2 \to \mathbb{RP}^\infty$ which factors through i as $f'\colon T^2 \to \mathbb{RP}^3 \to \mathbb{RP}^\infty$. It follows that $[f] = [f'] \in \operatorname{im}(i_*)$ and thus i_* is surjective.

Now it remains to show the injectivity of i_*. Take homotopy classes $[f_0], [f_1] \in [T^2, \mathbb{RP}^3]$. Up to homotopy, we may assume f_0 and f_1 are both CW maps. If $i_*([f_0]) = i_*([f_1])$, then there exists a homotopy map $H\colon [0,1] \times T^2 \to \mathbb{RP}^\infty$ such that $H|_{\{0\}\times T^2} = i \circ f_0$ and $H|_{\{1\}\times T^2} = i \circ f_1$. Applying the CW approximation theorem, relative to $\{0,1\} \times T^2$, the map H is homotopic to the CW map $H'\colon [0,1] \times T^2 \to \mathbb{RP}^\infty$. But $[0,1] \times T^2$ is a 3-dimensional CW complex, and hence $\operatorname{im}(H')$ is contained in the 3-skeleton of $\mathbb{RP}^\infty$, which is $\mathbb{RP}^3$. Therefore, H' is indeed a homotopy map between f_0 and f_1, proving that $[f_0] = [f_1]$. This completes the proof of our claim. □

Problem B.8. Let M be a closed manifold such that $H_1(M;\mathbb{Z}) = 0$. Consider the continuous map $f\colon M \to S^1$. Show that f cannot be a fibration.

Solution. Using the property of the Eilenberg–McLane space $S^1 \approx K(\mathbb{Z},1)$, we see $[-, S^1] \cong H^1(-, \mathbb{Z}) = 0$. It follows that f is homotopic to the zero map. Let $H\colon [0,1] \times M \to S^1$ be the homotopy map that deforms f to a constant map valued at b.

Assume f is a fibration. Consider the homotopy lifting problem

$$\begin{array}{ccc} M & \xrightarrow{\ \mathrm{id}\ } & M \\ \downarrow & \overset{\tilde{H}}{\nearrow} & \downarrow f \\ [0,1] \times M & \xrightarrow{\ H\ } & S^1. \end{array}$$

Here $\tilde{H}$ gives a homotopy from id to $\tilde{H}_{\{1\}\times M}\colon M \to M$. Note that $\operatorname{im}(\tilde{H}_{\{1\}\times M}) \subset f^{-1}(b)$; since all fibres of f are homotopic, they are all nonempty, which shows that f is not surjective. However, the map $\mathrm{id}\colon M \to M$ has degree 1 at each component, which cannot be homotopic to a non-surjective map with degree 0. This leads to a contradiction. □

PHOTOGRAPHS — APRIL 17, 2025; AT IKEA MALL WUKESONG, BEIJING. *There are no desks or chairs, but many mugs are available. Please be invited to describe the topology of these structures.*

REFERENCES

[Hat02] Allen Hatcher. *Algebraic topology*. Cambridge University Press, 2002. Online version available at https://pi.math.cornell.edu/~hatcher/AT/AT+.pdf.

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