Plancherel formula for spherical functions

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$\mathcal{O} \subset F$: nonarch ~~field~~ local field.

G split red grp,

X spherical var $\circlearrowleft G$.

§ Abstract Plancherel decomp

Fix $G(F)$-eigenmeasure on $X(F)$, eigenchar η.

$$\underset{\substack{\circlearrowleft \\ G(F)\text{-unitarily} \\ (\text{twist by } \sqrt{\eta})}}{L^2(X(F))} = \underset{\text{"direct integral"}}{\int^{\oplus}} \pi_\lambda \, d\mu_{Pl}(\lambda)$$

where $\lambda \in$ unitary dual of $G(F)$

π_λ : λ-isotypic part of $L^2(X(F))$.

$$\|\phi\|^2_{L^2} = \int \|\underset{\substack{\uparrow \\ \text{image of } \pi_\lambda}}{\phi_\lambda}\|^2_{\pi_\lambda} \, d\mu_{Pl}(\lambda) .$$

Unramified part [Sak 13] (+ assumptions on X)

$$\underset{\substack{\cup \\ e := \mathbb{1}_{X(\mathcal{O})} \text{ basic} \\ \text{vector when } X \text{ is smooth}}}{L^2(X(F))^{G(\mathcal{O})}} = \int^{\oplus}_{\check{A}^{(1)}_X/W} \pi_t \, \underbrace{d\mu_{Pl}(t)}_{\mu(t)\,dt} \quad \text{via classical Satake.}$$

$\mu(t)\,dt \leftarrow$ Want to understand this density (dt = "usual measure").

Goal Understand $\mu(t)$. characterized by

$$\langle T_V e, T_W e\rangle_{L^2} = \int_{\check{A}/W} \chi_V(t) \overline{\chi_W(t)} \, \mu(t)\, dt, \quad \forall\, V, W \in \operatorname{Rep} \check{G}.$$

It suffices to consider $\langle T_V e, e\rangle_{L^2}$, $\forall\, V \in \mathrm{Rep}\, \check{G}$.

<u>Slogan</u> $\underset{\circlearrowleft \check{G}}{\mathbb{PL}_X} := \underline{\mathrm{End}}_{\mathrm{Hecke}}(\delta_X)$ categorifies the (unram) Plancherel meas.

$\rightsquigarrow \mathbb{PL}_X$: dga $/ \mathcal{O}(\check{\mathfrak{g}}^*)^{\natural}$ + $\check{G}$-action.

<u>Tool</u> $[\cdot] :=$ tr Frob, $\quad [\mathrm{Hom}_{\mathrm{Shv}(X_F/G_O)}(T_V * \delta_X, \delta_X)^{\vee}]$

$$\| \ (8.13)$$

$$[\mathrm{Hom}_{\check{G}\text{-mod}}(V, \mathbb{PL}_X)^{\vee}].$$

<u>Next</u> Try to deduce a formula for μ from local conj.

<u>§ Setup</u>

$\mathbb{F}_q$, $p \mid q$, $k = \bar{\mathbb{Q}}_\ell$ $(\ell \neq p)$. Fix $k \simeq \mathbb{C}$.

$F := \mathbb{F}_q((t)) \supseteq \mathcal{O} = \mathbb{F}_q[\![t]\!]$.

$q^{1/2} \in k$, $\check{G}, \check{M} / k$.

$M = T^*(X, \Psi)$ $\quad$ (Ψ usually trivial)

$\quad$ hyperspherical $/ \mathbb{F}_q$, X spherical.

$\check{M} = V_X \times^{\check{G}_X} \check{G}$

$V_X = S_X \oplus (\mathfrak{g}_X^{\perp} \oplus \mathfrak{g}_e^{\vee})$.

(h, e, f) : SL_2-triple in $\mathfrak{g}$, $\mathfrak{g} \simeq \mathfrak{g}^*$.

<u>Require</u> For all simple spherical root of T,

$$\mathring{X} P_\alpha / R_u(P_\alpha) \simeq SO_{2n} \backslash SO_{2n+1}.$$

SO_{2n} split $\Rightarrow$ all colors are $/\mathbb{F}_q$.

$\mathrm{Gal}_f \circlearrowright S_x$ "simple".

Have $X \simeq S^+ \times^{HU} G$, $H \subset G$ reductive, S^+ rep of H.

modular char of $H \circlearrowright S^+$ extends to $\eta : G(f) \to \mathbb{R}^\times_{>0}$.

$\Rightarrow$ $X(f)$ has a $(G(f), \eta)$-eigenmeasure.

<u>Normalization</u> $\mathrm{vol}(X(\mathcal{O})) = \dfrac{|X(\mathbb{F}_q)|}{|G(\mathbb{F}_q)|} \cdot q^{\dim G - \dim X}$.

$G(f) \circlearrowright L^2(X(f))$ unitarily, right regular + $\sqrt{\eta}$-twist

$V_x = \underset{\substack{\circlearrowleft \\ \mathbb{G}_m}}{S_x} \oplus (\mathfrak{g}_x^{\perp} \cap \check{\mathfrak{g}}_x)$ as a $\check{G}_x \times \mathbb{G}_m$-variety. (c.f. §4.5)

all degs are > 0.

Fix $L(x) \subset P(x) =$ parabolic $\subset G$.

<u>Prop</u> Assume $\mathbb{PL}$-conj 8.1.8 ($\Leftarrow$ Local conj),

then $\forall f \in C_c(\breve{A} /\!\!/ W)$,

$$\mu(f) = \underset{\substack{\uparrow \\ \text{little Weyl}}}{|W_x|^{-1}} \cdot \int_{\breve{A}_x^{(1)}} f(\underbrace{q^{-\rho_L(x)}}_{\substack{\uparrow \\ \text{the cochar } -2\rho_L(x) \\ \text{evaluated at } q^2}} t) \cdot \frac{|\det(1 - \mathrm{Ad}(t) \mid \check{\mathfrak{g}}_x / \check{\mathfrak{a}}_x)|}{\det(1 - (t, q^{1/2}) \mid V_x)} \cdot \underset{\substack{\uparrow \\ \text{prob Haar meas on } \breve{A}_x^{(1)}}}{dt}$$

<u>Proof</u> By def, $\langle T_v e, e \rangle_{L^2} = \int_{X(\mathcal{O})} \int_{G(f)} \sqrt{\eta(g)} \cdot T_v(g) \cdot \underbrace{\mathbb{1}_{X(\mathcal{O})}(xg)}_{e(xg)} \, dg \, dx$

tr Frob ($\mathbb{PL}$-conj)

$\underline{(8.13)}$ $[\mathbb{PL}_x^{(M,v)}] \overset{\downarrow}{=} [\mathrm{Hom}(V, \mathcal{O}_{\check{M}}^{\hbar})^\vee]$.

"analytic shearing" (no η involved, see §6.8.1).

<u>Claim</u> $\mathrm{Hom}(V, \mathcal{O}_{\check{M}})^\vee = (V \otimes \mathcal{O}_{\check{M}})^{\check{G}} = (V \otimes \mathrm{Sym}\, V_x)^{\check{G}_x}$ ignoring $\square$.

pf Here: $\mathcal{O}_{\check{M}}$, Sym are in IndRep $\check{G}$ where
duality $\vee$ on extends from Rep $\check{G}$ to it.

$\Rightarrow$ the 1st equality: easy.

The 2nd equality:

$$\mathcal{O}_{\check{M}} = (\mathcal{O}_{\check{G}} \otimes \mathrm{Sym}\, V_x^\vee)^{\check{G}_x} \Rightarrow \mathcal{O}_G^\vee \simeq \mathcal{O}_G$$

$$\mathcal{O}_{\check{M}}^\vee = (\mathcal{O}_{\check{G}} \otimes \mathrm{Sym}\, V_x)^{\check{G}_x} =: {}^{alg}\mathrm{Ind}_{\check{G}_x}^{\check{G}} (\mathrm{Sym}\, V_x).$$

$$\Rightarrow V \otimes \mathcal{O}_{\check{M}}^\vee = {}^{alg}\mathrm{Ind}_{\check{G}_x}^{\check{G}} (V \otimes \mathrm{Sym}\, V_x)$$

$$\Rightarrow (V \otimes \mathcal{O}_{\check{M}}^\vee)^{\check{G}} = (V \otimes \mathrm{Sym}\, V_x)^{\check{G}_x}$$

Get the claim.

("alg" induction of $\check{G}$-modules: [Grosshans, LNM, 1673]).

Moreover, $\mathbb{G}_m \curvearrowright$ RHS of claim $= (V \otimes \mathrm{Sym}\, V_x)^{\check{G}_x}$

via $\mathbb{G}_m \curvearrowright V_x \overset{\square}{\rightsquigarrow} q^{-1/2}$

$2\rho_{L(x)} \curvearrowright V \rightsquigarrow q^{-\rho_{L(x)}}$.

$\langle T_V e, e\rangle_{L^2} = \cdots = \sum_{a,b}$ tr Frob acting on $\mathbb{G}_m \curvearrowright V$ of deg $= a$

$\mathbb{G}_m \curvearrowright V_x$ of deg $= b$

$$= \sum_{a,b} \int_{U_x} \chi_{V^a}(g) \cdot \mathrm{tr}(g \mid \mathrm{Sym}^b V_x) \cdot q^a \cdot q^{-b/2}\, dg$$

+ Weyl unitarian trick.

where $U_x \subset \check{G}_x$ max cpt subgrp.

$$= \int_{U_x} \chi_V(q^{-\rho_{L(x)}} g) \cdot \mathrm{tr}((g, q^{-1/2}) \mid \mathrm{Sym}\, V_x)\, dg.$$

Use Weyl's integral formula to get

$$|W_x|^{-1} \cdot \int_{\check{A}_x^{(1)}} (\cdots) \underbrace{\left| \det (1 - \mathrm{Ad}(t) \mid \check{\mathfrak{g}}_x / \check{\mathfrak{a}}_x) \right|}_{\text{Weyl denominator}} \cdot dt$$

$\rightsquigarrow$ vary V to conclude. $\square$

§ Known computations of μ

X affine spherical s.t. T^*X hyperspherical

X homogeneous / $\check{G}_X = \check{G}$ "strongly tempered".

+ combinatorial assumption.

$\Rightarrow$ μ = the expected.

Ref [Sak 13], [Sakellaridis - Wang]

allow some non-sm X,

e = "IC-func" of X.

In these cases, μ = that in the Prop

but $V_X \rightsquigarrow V_X'$: $\check{A}_X' \times \mathbb{G}_m$ whose {wts} $\subset X^*(\check{A}_X)$ is W_X-inv.

Prop 9.3.3 $V_X' = S_X' \oplus V_X''$ can be matched with

$V_X = S_X \oplus (\check{\mathfrak{g}}_X^{\perp} \oplus \check{\mathfrak{g}}_e)$

under various assump'ns.

Rmk 9.3.4 Include all sm affine spherical X that BZSV know about.

Pf Use [Sak13] and [SWa]. $\square$

Ingredient : Rmk 9.3.6

$X = pt$, $\check{G}_X = \{1\}$,

$vol(X(\mathcal{O})) = q^{\dim G}/|G(\mathbb{F}_q)|$.

(h, e, f) = principal $\mathfrak{sl}_2$-triple.

$S_X = \{0\}$, $\check{\mathfrak{g}}_e$ = Borel $\check{\mathfrak{b}}$, U = max unip in $\check{G}$.

Formula of Steinberg [Gross, Prop 4.7, 1997]

$\mathrm{vol}(X(\mathcal{O})) = \det(1-q^{-1}V)^{-1}$

where V = tangent space at 0 of $\check{\mathfrak{t}}^* /\!/ W =: \mathfrak{c}$.

$\overset{\shortparallel}{\bigoplus_{d\geq 0}} V_d$, V_d = prim inv of deg d.

$\check{M} = \check{\mathfrak{u}} \times \check{G}$, $\quad \check{M}/G = \check{\mathfrak{u}} \xleftarrow[\text{tangent at any pt}]{\sim} f + \check{\mathfrak{u}} \xrightarrow[\text{Kostant's section}]{\sim} \mathfrak{c}$

$\hookrightarrow \check{\mathfrak{u}} \simeq V$.

$f + \check{\mathfrak{u}} \hookrightarrow \underset{\check{\mathfrak{g}}^*}{\overset{\check{\mathfrak{g}}}{\wr\!\shortmid}} \twoheadrightarrow \check{\mathfrak{g}}^* /\!/ G$.

<u>Amusing exercise</u> $\forall t \in \mathbb{G}_m$, $t^2 \curvearrowright V$ corresponds to $t^{(2+(h\text{-weight}))} \curvearrowright \check{\mathfrak{g}}_e = \check{\mathfrak{u}}$.

This implies $\det(1-q^{-1}V)^{-1} = (1-q^{-1/2} \mid \check{\mathfrak{g}}_e)^{-1}$.

§ Hecke module structure

alg ver of $L^2(X(F))^{G(\mathcal{O})}$: $\mathcal{C}_c^\infty(X(F), k)^{G(\mathcal{O})}$.

$$\begin{array}{ccc}
\mathrm{SHV}(X_F/G_{\mathcal{O}}) & \xrightarrow[\text{Local conj}]{\sim} & QC^{!!}(\check{M}/\check{G}) \\
\textcircled{?} \Big\downarrow & & \Big\downarrow \text{categorical tr of } F = \text{Frob} \\
\underbrace{\mathcal{C}_c^\infty(X(F); k)^{G(\mathcal{O})}}_{} & \xrightarrow[\text{isom}/k]{\sim} & k[\underbrace{\Delta_{\check{M}/G} \cap \mathrm{Graph}_F}_{}]
\end{array}$$

derived one: $\forall G(\mathcal{O})x$ contributes to $H^\bullet(\underbrace{G_x(\mathcal{O})}_{}; k)$ $\overset{\wr\!\shortmid}{\mathbb{C}}$

only deg $=0$ $\Leftarrow$ Profinite.

$\overset{\shortparallel}{\{(g,m) \in \check{G} \times M \mid gm = Fm\}/\check{G}}$

$h(g,m) = (hgh^{-1}, hm)$, $h \in \check{G}$.

For ③ : [Zhu 18] the group case

cf. In [AGKRRV 20b, §15, "Trace conj"].

Rmk $k[\Delta_{\check{M}/G} \cap \mathrm{Graph}_F] \simeq k[\{(g,v) \in \check{G}_X \times V_X \mid gv = q^{1/2}v\}]^{\check{G}_X}$.

☺ $\check{M} = V_X \times^{\check{G}_X} \check{G}$ + effect of $\square$.

Final remark For categorical traces / TQFT in BZ, Between the electric-magnetic duality & Langlands, Chap 12.