Introduction to global conjecture
March 4

§1 Introduction & motivation

<u>Setup</u> $\mathbb{F} = \mathbb{F}_q$ & $k = \bar{\mathbb{Q}}_\ell$ $(\ell \neq p)$ or $\mathbb{F} = k = \mathbb{C}$. Fix $\bar{\mathbb{Q}}_\ell \simeq \mathbb{C}$, $\sqrt{q} \in k$.

Σ sm proj curve $/\mathbb{F}$.

G split reductive grp $/\mathbb{F}$

M hyperspherical G-var + polarization

($\mathbb{G}_{gr} \curvearrowright M$, Hamiltonian sm affine, etc.)

$\rightsquigarrow$ $\check{G}$ $/k$, $\check{M}$ hypersph $\check{G}$-var.

<u>Recall</u> Primary example of relatie duality:

$X, \check{X}$ sph var, $M = T^*X$, $\check{M} = T^*\check{X}$,

$$(G \curvearrowright T^*X) \longleftrightarrow (\check{G} \curvearrowright T^*\check{X})$$

$$X = H \backslash G \qquad\qquad \check{X} = \text{std}$$

e.g. $\quad \mathbb{G}_m \backslash GL_2 \qquad\qquad \mathbb{A}^2$.

<u>Numerical version</u>

Period formula:

Automorphic X-period = Spectral $\check{X}$-period

e.g. $X = H \backslash G$, $\phi : G(F) \backslash G(\mathbb{A}) \to \mathbb{C}$ autom form

$\sigma_\phi : \mathrm{Gal}_F \to \check{G}$ parameter of ϕ

<u>Then</u> $$\int_{[H]} \phi(h)\, dh = \sum_{x \in (\check{X})^{\sigma_\phi}} L(T_x)$$

L-value of tangent complex

Categorical version

Conj (Local, revisit)

$$\mathrm{Shv}(LX/L^+G) \simeq QC^!(\underbrace{T[-1]\check{X}}/\check{G})$$

derived self-inter'n of $\check{X}$ (c.f. HKR).

Conj (Global geom, ignoring normalizations)

Geom Langlands corr (de Rham / Betti / restr)

$$\mathrm{Shv}(\mathrm{Bun}_G(\Sigma)) \simeq QC^!(\mathrm{Loc}_{\check{G}}(\Sigma))$$

$$\mathcal{P}_X(\Sigma) \longleftrightarrow \mathcal{L}_{\check{X}}(\Sigma)$$

period sheaf — L-sheaf

[BZSV, §10] — [BZSV, §11].

(after projection to $\mathcal{N}$ nilp sing supp).

Key $\mathcal{P}_X$ & $\mathcal{L}_{\check{X}}$ recover period & L-funcs via $\mathrm{Tr}(\mathrm{Frob} \mid H^*_{\acute{e}t}(\Sigma, -))$.

$$\text{Geom} \xrightarrow{\text{Frob trace}} \text{Arith}$$

$$\text{Local} \xrightarrow{\text{Factorization / Euler prod}} \text{Global}$$

§2 $\mathrm{Bun}_G^X(\Sigma)$

Heuristic construction: To understand $G \curvearrowright X$,

think about the pull-push along the correspondence

$$\begin{tikzcd} & X/G \arrow[dl] \arrow[dr] & \\ pt/G & & pt = \mathrm{Spec}\,\mathbb{F} \end{tikzcd}$$

from pt to pt/G.

$$\begin{tikzcd}
 & \mathrm{Map}(\Sigma, X/G) \arrow[dl, "\pi"'] \arrow[dr] & \\
\mathrm{Map}(\Sigma, pt/G) & & \mathrm{Map}(\Sigma, pt) \\
\mathrm{Bun}_G(\Sigma) \arrow[u, equal] & & pt \arrow[u, equal]
\end{tikzcd}$$

$\rightsquigarrow$

<u>Definition</u> $\mathrm{Bun}_G^X(\Sigma) := \mathrm{Map}(\Sigma, X/G)$ alg stack $/\mathbb{F}$.

This also defines a moduli

$$\mathrm{Bun}_G^X : \Sigma \longmapsto \mathrm{Bun}_G^X(\Sigma) := \left\{ (P, s) \;\middle|\; \begin{array}{l} P \to \Sigma \text{ G-bundle on } \Sigma \\ s \in \Gamma(\Sigma, (\mathrm{coad}\, P)_G \otimes \Omega_X \otimes K^{1/2}) \\ \text{with a chosen } K^{1/2} \text{ on } \Sigma \end{array} \right\}$$

"Spin structure", to be explained later

<u>Upshot</u> It suffices to understand $\pi_! \underline{k}_{\mathrm{Bun}_G^X(\Sigma)}$ along

$$\pi : \mathrm{Bun}_G^X(\Sigma) \longrightarrow \mathrm{Bun}_G(\Sigma).$$

<u>Towards geometry of π</u>

(1) When $X = H \backslash G$, $\mathrm{Bun}_G^X = \mathrm{Bun}_H$ ($\Rightarrow$ smooth)

$\pi : \mathrm{Bun}_H \to \mathrm{Bun}_G$ is an analog of $[H] \hookrightarrow [G]$.

(2) For $\mathcal{E} \in \mathrm{Bun}_G(\Sigma)$, $\pi^{-1}(\mathcal{E}) \simeq \Gamma(\Sigma, \mathcal{E} \otimes K^{1/2})$.

(3) Have a pullback

$$\begin{tikzcd}
\mathrm{Map}(\Sigma, X/G) = \mathrm{Bun}_G^X(\Sigma) \arrow[r] \arrow[d, "\pi"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{Map}(\Sigma, X/(G \times \mathbb{G}_{gr})) \arrow[d] \\
\mathrm{Map}(\Sigma, pt/G) = \mathrm{Bun}_G(\Sigma) \arrow[r, "\mathrm{id} \boxtimes K^{1/2}"] & \mathrm{Bun}_{G \times \mathbb{G}_{gr}}(\Sigma) = \mathrm{Map}(\Sigma, pt/(G \times \mathbb{G}_{gr}))
\end{tikzcd}$$

(4) Harder-Narasimhan stratification:

$$\mathrm{Bun}_G = \coprod_{\nu \in \mathrm{Hom}(G, \mathbb{G}_m)^\vee} \mathrm{Bun}_G^\nu$$

each $\mathrm{Bun}_G^\nu(\Sigma) \hookrightarrow \mathrm{Bun}_G(\Sigma)$ clopen & smooth
(in 2-cat of Artin stacks / $\mathbb{F}$).

Can choose $\begin{matrix} X & \hookrightarrow & \mathbb{A}^n \\ \circlearrowleft & & \circlearrowleft \\ G & & GL_n \end{matrix}$ to reduce to $X = \mathbb{A}^n$, $G = GL_n$

$\rightsquigarrow$ $\mathrm{Bun}_n^{\mathbb{A}^n} \longrightarrow \mathrm{Bun}_n$ is tame as a global quotient of morph of schs on Bun_n^ν.

<u>Rmk</u> See on spec side that $\mathrm{Loc}_{\check{G}}$ is singular,
e.g. for $\Sigma = \mathbb{P}^1$, $\mathrm{Loc}_{\check{G}}(\mathbb{P}^1) = B\check{G} \times_{\check{\mathfrak{g}}/\check{G}} B\check{G} = \Omega(\check{\mathfrak{g}}/\check{G})$
loop quotient space, as a derived sch.

Can also regard Bun_G^X as a derived stack.
$\rightsquigarrow$ Everything has a derived nature.
But we ignore this
b/c $\mathcal{P}_X \longleftrightarrow \mathcal{L}_{\check{X}}$ only sensitive to topology.

<u>§3 Formula of period sheaf</u>

Let $M = T^*X$, X hypersph G-var / $\mathbb{F}$,

<u>Technical ass'n</u> X has an eigenmeasure
$\Rightarrow$ canonical bundle of $[X/(G \times \mathbb{G}_{gr})]$ (pulled back from $B(G \times \mathbb{G}_{gr})$)
is specified by $(\eta_X, \gamma_X) : G \times \mathbb{G}_{gr} \longrightarrow \mathbb{G}_m$.

$\rightsquigarrow$ Get $\deg_{\eta_X} : \mathrm{Bun}_G(\Sigma) \xrightarrow{\eta_X} \underset{\substack{\| \\ \text{Picard groupoid}}}{\mathrm{Bun}_{\mathbb{G}_m}(\Sigma)} \xrightarrow{\deg} \mathbb{Z}$

In this case, by Grothendieck-Riemann-Roch,
$$\dim \mathrm{Bun}_G^X(\Sigma) = \underbrace{(g-1)(\dim G - \dim X + \gamma_X)}_{=:\beta_X} + \deg_{\eta_X} \qquad (g = \mathrm{genus}(\Sigma))$$

(Explanation: $\dim G - \dim X \leftrightarrow G \curvearrowright X$ (to appear later)
$\gamma_X \longleftrightarrow \mathbb{G}_{gr} \curvearrowright X$ by scaling.)

Recall We are interested in the pull-push along

$$\begin{tikzcd} & \mathrm{Bun}_G^X(\Sigma) \arrow[dl, "\pi"'] \arrow[dr] & \\ \mathrm{Bun}_G(\Sigma) & & pt \end{tikzcd}$$

Define period sheaf (normalized)

$$\mathcal{P}_X^{(norm)} := \pi_! \underline{k}_{\mathrm{Bun}_G^X(\Sigma)}[\dim \mathrm{Bun}_G^X(\Sigma)] \in \mathrm{Shv}(\mathrm{Bun}_G(\Sigma))$$

& its "dual sheaf"

$$\mathcal{P}_X^* \text{ "}= \mathbb{D}\mathcal{P}_X\text{" } := \pi_* \omega_{\mathrm{Bun}_G^X(\Sigma)}[-\dim \mathrm{Bun}_G^X(\Sigma)] \in \mathrm{Shv}(\mathrm{Bun}_G(\Sigma)).$$

These are Weil sheaves, i.e. Frob-equiv when $\mathbb{F} = \overline{\mathbb{F}}_q$.

§4 Period function

Slogan $\mathrm{Tr}(\mathrm{Frob}_q \mid H^*_{geom}(X, \mathcal{P}_X))$ recovers period func

$$P_X : \mathrm{Bun}_G(\mathbb{F}_q) \longrightarrow k.$$

Spin structure K = canonical bundle on Σ.

Let $(n_v)_v$ be even integers with $\sum n_v = 2g-2$.

$\rightsquigarrow K^{1/2} := \mathcal{O}(\sum_v \frac{n_v}{2} \cdot v)$.

This appears in

- $\mathcal{A}_G$-side: moduli of $\mathrm{Bun}_G^X(\Sigma)$,
- $\mathcal{B}_{\check{G}}$-side: "canonical" formula of L-sheaf

$$\mathcal{L}_{\check{X}} = \pi_*^{II}(\omega_{\mathrm{Loc}_{\check{G}}^{\check{X}}(\Sigma)} \otimes K^{-1/2}).$$

where $\pi_*^{\square}: \mathrm{IndCoh}^{\square}(\underbrace{\mathrm{Loc}_{\check{G}}^{\check{X}}(\Sigma)}_{\mathrm{Map}(\Sigma_{\mathrm{Betti}},\, \check{X}/\check{G})}) \longrightarrow \mathrm{IndCoh}(\mathrm{Loc}_{\check{G}}(\Sigma))$.

Let F = func field of Σ.

Have uniformization $\quad \mathrm{Bun}_G(\mathbb{F}_q) = G(F) \backslash G(\mathbb{A}_F) / G(\hat{\mathcal{O}})$.

<u>Example</u> For $G = \mathbb{G}_m$, $\quad \mathrm{Bun}_{\mathbb{G}_m}(\mathbb{F}_q) = F^\times \backslash \mathbb{A}_F^\times / \prod_v \mathcal{O}_v^\times$

$$\mathcal{O}(x) \longleftrightarrow \varpi_x \in \mathbb{G}_m(\mathbb{A}), \qquad x \in \Sigma$$

$$\mathcal{K}^{1/2} \longleftrightarrow \prod \varpi_v^{n_v/2} =: \partial^{1/2} \in \mathbb{G}_{gr}$$

<u>Fact</u> Take $g \in G(\mathbb{A}_F) \longleftrightarrow g \in \mathrm{Bun}_G(\mathbb{F}_q)$.

Then $\quad (\pi^{-1}(g))(\mathbb{F}_q) = X(F) \cap \prod_v X(\mathcal{O}_v) \cdot (g^{-1}, \partial^{-1/2})$

$\circlearrowleft$ $G \times \mathbb{G}_{gr}$.

The formula of period func:

$$P_X : \mathrm{Bun}_G(\mathbb{F}_q) = G(F) \backslash G(\mathbb{A}) / G(\hat{\mathcal{O}}) \longrightarrow k$$

$$g \longmapsto \sum_{x \in X(F)} (g, \partial^{1/2}) \cdot \Phi(x)$$

Here $\Phi(x) \in \mathcal{S}(X)$ "basic" char func

$= \mathbb{1}_{X(\mathcal{O})}$ if X smooth

$$(g, \lambda)\Phi(x) = \Phi(x(g, \lambda)).$$

<u>Next talk</u> Introduce a normalization

$$P_X^{norm}(g) := q^{-\beta_X/2} \cdot \sum_{x \in X(F)} g \star (\partial^{1/2} \cdot \Phi(x))$$

$$= q^{\frac{g-1}{2}(\dim X - \dim G)} \sum_{x \in X(F)} \underbrace{(g, \partial^{1/2}) \star \Phi(x)}_{|\eta_X(g)|^{1/2} \cdot |\partial^{1/2}|^{\gamma_X/2} \cdot (g, \partial^{1/2}) \cdot \Phi(x)}.$$

§5 Towards numerical Conjecture

<u>Philosophy</u> $\mathcal{P}_X \longleftrightarrow \mathcal{L}_X^\vee \xrightarrow{\text{Frob trace}}$ period formula for $\langle \mathcal{P}_X, \varphi^\vee \rangle$

(geom) (arith)

Here $\mathcal{P}_X$ = period func, φ = autom form.

e.g. $X = H\backslash G$, $\langle \mathcal{P}_X, \varphi^\vee \rangle = \int_{[H]} \varphi(h)\, dh$.

<u>Ingredients</u>

(1) Grothendieck-Lefschetz trace formula:

<u>Arith setting</u> If X var $/\mathbb{F}_q$, then

$$\# X(\mathbb{F}_{q^n}) = q^{\dim X} \cdot \mathrm{Tr}(\underset{\substack{\uparrow \\ \text{geom Frob}}}{\mathrm{Frob}_q^n} \mid \underbrace{H_c^*(X, \bar{\mathbb{Q}}_\ell)}_{\substack{\text{is} \\ \pi_! \bar{\mathbb{Q}}_\ell}}) \qquad (\ell \neq p).$$

$\pi : X \to pt$

<u>General setting</u> Replace $H_c^*(X, \bar{\mathbb{Q}}_\ell)$ with certain $H^*_{geom}(X)$.

<u>Input</u> Tech of [Gaitsgory-Lurie] (dealing with Bun_G), using

$$\begin{tikzcd} X \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & X/G \arrow[d] \\ pt = \mathrm{Spec}\, \mathbb{F} \arrow[r] & BG \end{tikzcd},$$

get
$$\begin{aligned} &\mathrm{Tr}(\mathrm{Frob}_q \mid H^*_{geom}(X/G)) \\ &= \mathrm{Tr}(\mathrm{Frob}_q \mid H^*_{geom}(BG)) \cdot \mathrm{Tr}(\mathrm{Frob}_q \mid H^*_{geom}(X)) \\ &= \frac{q^{\dim G}}{|G(\mathbb{F}_q)|} \cdot \frac{|X(\mathbb{F}_q)|}{q^{\dim X}} \end{aligned}$$

This is compatible w/ automorphic normalization of [BZSV, §9].

- Have $X \simeq S^+ \times^{HU} G$, $H \subset G$ reductive subgrp.

  U a max unipotent,

  S^+ rep of H.

- If modular char of $H \curvearrowright S^+$ extends to $\eta: G(\mathbb{F}) \to \mathbb{G}_m$ then $X(\mathbb{A})$ has a $(G(\mathbb{F}), \eta)$-eigenmeasure.

Require that
$$\mathrm{vol}(X(\mathcal{O})) = q^{\dim G - \dim X} \cdot \frac{|X(\mathbb{F}_q)|}{|G(\mathbb{F}_q)|}.$$

(2) From Tr(Frob) to L-function.

<u>Linear algebra</u> $V \in \mathrm{Vect}_{\mathbb{C}}$, $T \in \mathrm{End}(V)$ with spectrum $\subset$ unit disk.

Then
$$\mathrm{Tr}(T : \mathrm{Sym}\, V \to \mathrm{Sym}\, V) = \det(1-A)^{-1},$$
$$\mathrm{Tr}(T : \mathrm{Sym}(V[1]) \to \mathrm{Sym}(V[1])) = \det(1-A).$$

<u>L-func'n</u> For $V \in \mathrm{Rep}(\check{G})$,
$$L(\varphi, V, s) = \prod (\text{char poly of conj class in } \check{G} \curvearrowright V).$$

<u>Fact</u>
$$L(\mathcal{T}_x) = \mathrm{Tr}(\mathrm{Frob}_q \mid \underbrace{\mathrm{Sym}\, H^1_{\acute{e}t}(\Sigma_{\bar{\mathbb{F}}_q}, T_x \check{X})[-1]}_{= (\mathcal{L}_{\check{X}})_P,\ P \in \mathrm{Loc}_{\check{G}}(\Sigma)}), \qquad x \in \check{X}.$$

Here $\mathcal{L}_{\check{X}}$ called L-sheaf, [BZSV, Chap 11]

taking Frob trace on $\mathcal{L}_{\check{X}}$ recovers the L-func.

<u>Note</u> Motivated to consider
$$\mathcal{D}(\mathrm{Bun}_G) \longrightarrow \mathrm{Vect}\ (\text{or } \mathcal{D}(\mathrm{Vect}))$$
$$\mathcal{T}_x \longmapsto \mathrm{Sym}\, H^1_{\acute{e}t}(\Sigma_{\bar{\mathbb{F}}_q}, T_x \check{X})[-1]$$

This is a shadow of Geom Langlands Conj.

(3) Sheaf-function dictionary:

<u>Lem 2.4.1</u> $\mathcal{F}, \mathcal{G}$ Weil shv on $X / \mathbb{F}_q$.

$f, \check{g}$ = trace func'ns of $\mathcal{F}$, $\mathbb{D}\mathcal{G}$,
$\mathbb{D}$ = Verdier duality.

Then $\displaystyle\sum_{x \in X(\mathbb{F}_q)} f(x)\check{g}(x) = \mathrm{Tr}(\mathrm{Frob}_q \mid \mathrm{Hom}(\mathcal{F}, \mathcal{G})^\vee)$.

<u>Application</u> Conditionally, for $\rho \in \mathrm{Loc}_{\check{G}}(\Sigma)$ sm point

$$\begin{array}{ccc} \rightsquigarrow \delta_\rho \in \mathrm{IndCoh}_{\mathcal{N}}(\mathrm{Loc}_{\check{G}}(\Sigma)) & & \text{skyscraper sheaf} \\ \big\downarrow \quad\quad \wr\big\downarrow \text{GLC (de Rham)} & & \\ \mathcal{F}_\rho \in D(\mathrm{Bun}_G) & & \text{Hecke eigensheaf} \end{array}$$

Get φ autom form from δ_ρ.

Take $\mathcal{F} = \mathcal{L}_{\check{X}}$, $\mathcal{G} = \delta_\rho$ in shv-func dictionary.

$$\begin{aligned} \langle P_X, \varphi^\vee \rangle &= \mathrm{Tr}(\mathrm{Frob} \mid \mathrm{Hom}(\mathcal{L}_{\check{X}}, \delta_\rho)^\vee) \\ &= q^{-(g-1)\dim G}\, \mathrm{Tr}(\mathrm{Frob} \mid \bigoplus_{x \in (\check{X})^\rho} \Lambda^* H^1(\rho, T_x \check{X})) \\ &= q^{-(g-1)\dim G} \sum_{x \in (\check{X})^\rho} L(T_x). \end{aligned}$$

<u>Next time</u> Explicit examples + computation of P_X^{norm}.