Examples of period sheaves & period functions
March 11

<u>Setup</u> $\mathbb{F} = \overline{\mathbb{F}}_q$ & $k = \overline{\mathbb{Q}}_\ell$ $(\ell \neq p)$ or $\mathbb{F} = k = \mathbb{C}$. Fix $\overline{\mathbb{Q}}_\ell \simeq \mathbb{C}$, $\sqrt{q} \in k$.

Σ sm proj curve $/\mathbb{F}$, F = func field of Σ

G split reductive grp $/\mathbb{F}$

M hyperspherical G-var + polarization.

$\rightsquigarrow$ $(G, M)/\mathbb{F}$, $(\check{G}, \check{M})/k$.

<u>Last time</u> (1) $\mathrm{Bun}_G^X(\Sigma) := \mathrm{Map}(\Sigma, X/G)$

carrying all info about $G \curvearrowright X$.

<u>Correction</u> The correct moduli should be

$$\mathrm{Bun}_G^X(\Sigma) = \{(\mathcal{E}, s) \mid \mathcal{E} \in \mathrm{Bun}_G(\Sigma),\ s \in \Gamma(\Sigma, P \overset{G}{\times} X \otimes \mathcal{K}^{1/2})\}$$

(2) Categorical global conj:

$$\begin{array}{cccc} \text{period sheaf} & \mathcal{P}_X & \longleftrightarrow & \mathcal{L}_{\check{X}} & \text{L-sheaf} \\ & \Big\downarrow & \text{Tr of Frob} & \Big\downarrow & \\ \text{period func} & P_X & & L_X & \text{L-func} \end{array}$$

Have seen a numerical compatibility.

(3) Known formulae:

$$\mathcal{P}_X^{\text{norm}} := \pi_! \underline{k}_{\mathrm{Bun}_G^X(\Sigma)} [\dim \mathrm{Bun}_G^X(\Sigma)] \in \mathrm{Shv}(\mathrm{Bun}_G(\Sigma))$$

$$\mathcal{L}_{\check{X}} := \pi''_{*}(\omega_{\mathrm{Loc}_{\check{G}}^{\check{X}}(\Sigma)} \otimes \mathcal{K}^{-1/2}) \quad \text{(to appear in §11)}.$$

$$P_X : \mathrm{Bun}_G(\mathbb{F}_q) = G(F) \backslash G(\mathbb{A}) / G(\hat{\mathcal{O}}) \longrightarrow k$$

$$\mathcal{E} \longleftrightarrow g \in G(\mathbb{A}) \longmapsto \sum_{x \in X(F)} (g, \partial^{1/2}) \underset{\shortparallel}{\Phi}(x)$$

$\delta(x)$ basic.

Today Get to know Bun_G^X, $\mathcal{P}_X$, $\mathcal{P}_X^{norm}$ via examples.

§1 Overflow about Tr(Frob)

Upshot Counting points of the groupoid $\mathrm{Bun}_G(\mathbb{F}_q)$ (not only a set).

Have that

$$|\mathrm{Bun}_G(\Sigma)(\mathbb{F}_q)| = \sum_{\mathcal{G}} \frac{1}{|\mathrm{Aut}(\mathcal{G})|}, \quad \mathcal{G} = \text{isom class of } G\text{-bundle}$$

On the other hand

$$|\mathrm{Bun}_G(\Sigma)(\mathbb{F}_q)| = q^{b_G} \cdot \mathrm{Tr}(\mathrm{Frob}_q \mid H^*(\mathrm{Bun}_G(\Sigma), \mathbb{Z}_\ell))$$

$$(b_G = \dim \mathrm{Bun}_G).$$

§2 Examples for point-counting

Dream Also want to count $|\mathrm{Bun}_G^X(\Sigma)(\mathbb{F}_q)|$

$\rightsquigarrow$ approach to a numerical duality result.

Easy case: Homogeneous

$X = H\backslash G$. $H \subset G$ reductive subgrp.

$$\Rightarrow \mathrm{Bun}_G^X = \mathrm{Bun}_H \xrightarrow{\pi} \mathrm{Bun}_G.$$

By abuse of notation write $g \in G(\mathbb{A}) \longleftrightarrow g \in \mathrm{Bun}_G(\mathbb{F}_q) = G(F)\backslash G(\mathbb{A})/G(\hat{\mathcal{O}})$.

Fact $\mathcal{P}_X : g \longmapsto \#(\text{ways of reductions from } g \in \mathrm{Bun}_G \text{ to an } H\text{-torsor})$

$\rightsquigarrow \quad \mathcal{P}_X(g) = |\pi^{-1}(g)|$

Then

$$|\mathrm{Bun}_G^X(\mathbb{F}_q)| = \sum_{g \in \mathrm{Bun}_G(\mathbb{F}_q)} \underbrace{|\pi^{-1}(g)| / |\mathrm{Aut}(g)|}$$

want a model of this sum.

Complicated case: Whittaker

Recall Whittaker setting:

$$\begin{array}{ccc} X = G/(U,\psi) & \longleftrightarrow & \check{X} = pt \\ G \circlearrowright \ \circlearrowleft \mathbb{G}_{gr} \text{ by } (1, e^{2\check{\rho}}) & & \check{G} \circlearrowright \ \circlearrowleft \mathbb{G}_{gr} \text{ trivially} \end{array}$$

Here $\psi : U \to \mathbb{G}_a$ fixed.

Known: $\gamma_X = \gamma_{\check{X}} = 0$, $\dim X = \dim G - \dim U$, $\dim \check{X} = 0$.

<u>Goal</u> Fix a "weight" $f : \mathrm{Bun}_G(\mathbb{F}_q) \to k$. To compute

$$\sum_{g \in \mathrm{Bun}_G(\mathbb{F}_q)} P_X(g) f(g) = \int_{G(F)\backslash G(\mathbb{A})} P_X(g) \cdot f(g)$$

Usage: for $f : g \mapsto 1/|\mathrm{Aut}(g)|$,

this sum concerns about Weil's conj for F.

Unnormalized period func $P_X : g \mapsto \sum_{x \in X(F)} (g, \partial^{1/2}) \cdot \Phi(x)$, $\Phi(x) \in \mathcal{S}(X)$ basic.

$$\begin{aligned} \Rightarrow \quad & \int_{G(F)\backslash G(\mathbb{A})} P_X(g) \cdot f(g) \\ &= \int_{G(F)\backslash G(\mathbb{A})} f(g) \sum_{x \in U(F)\backslash G(F)} (g, \partial^{1/2}) \cdot \Phi(x) \\ &= \int_{U(F)\backslash G(\mathbb{A})} f(g) (g, \partial^{1/2}) \cdot \Phi(g) \end{aligned}$$

<u>Need</u> (i) A formula to characterize $\partial^{1/2} \cdot \Phi(g)$.

(ii) A measure.

(i) [§3.4.5] $\Rightarrow$ $\partial^{1/2} \circlearrowright X = U\backslash G$ via scaling $a_0^{-1} := e^{2\check{\rho}}(\partial^{-1/2}) \in T(\mathbb{A}_F)$

$$\mathrm{Supp}(\partial^{1/2} \cdot \Phi) = U(\mathbb{A}) \cdot a_0 \cdot G(\hat{\mathcal{O}}).$$

Here $\partial^{1/2} \cdot \Phi$ is characterized by

$$(\partial^{1/2} \cdot \Phi)(\tilde{x} t) = \psi(t) \cdot (\partial^{1/2} \cdot \Phi)(\tilde{x}), \quad \tilde{x} \in \underset{\circlearrowleft \mathbb{G}_a}{\underset{\|}{\tilde{\Phi}(\mathbb{A})}}, \ t \in \mathbb{G}_a(\mathbb{A})$$

space of $\mathbb{A}^1$-bundle over X

(ii) Let du on $\mathcal{U}(\mathbb{A})$ s.t. $\mathrm{vol}(\mathcal{U}(\mathbb{O})) = 1$,

$$\mathrm{vol}(\mathcal{U}(F)\backslash\mathcal{U}(\mathbb{A})) = q^{(g-1)\dim \mathcal{U}}.$$

$\rightsquigarrow$ $d'u := d(a_0^{-1} u a_0) = |e^{2\rho}(a_0^{-1})| \cdot du = q^{-(g-1)\langle 2\rho, 2\check{\rho}\rangle} \cdot du$

(2ρ = sum of roots for $\mathcal{U}$).

s.t. $\mathrm{vol}(\mathcal{U}(F)\backslash\mathcal{U}(\mathbb{A})) = q^{(g-1)(\dim \mathcal{U} - \langle 2\rho, 2\check{\rho}\rangle)}$.

So the desired integral $= q^{(g-1)(\dim \mathcal{U} - \langle 2\rho, 2\check{\rho}\rangle)} \int_{\mathcal{U}(F)\backslash\mathcal{U}(\mathbb{A})} \psi(u) f(u a_0)\, d'u$

*<u>Check: Matching the formula for $\mathcal{P}_X^{\mathrm{norm}}$</u>

Recall
- $\beta_X := (g-1)(\dim G - \dim X + \gamma_X)$,
- For $X \simeq S^+ \times^{H\mathcal{U}} G$, $\gamma_X = \dim S^+ - \langle 2\rho, \varpi\rangle$,

  ϖ = char assoc to SL_2 for (G, M).

In homogeneous case, $X = H\backslash G$ with $\beta_X = b_H = (g-1)\dim H$.

In Whittaker case, $X = \mathcal{U}\backslash G$ with H, S^+ triv, $\varpi = 2\check{\rho}$,

$\Rightarrow$ $\beta_X = \beta_{\mathcal{U}\backslash G} = (g-1)(\dim \mathcal{U} - \langle 2\rho, 2\check{\rho}\rangle)$.

<u>Resulting constr</u> $\mathcal{P}_X^{\mathrm{norm}}(f) = q^{-\beta_X/2} \cdot q^{\beta_X} \cdot W(f) = q^{\beta_X/2} \cdot W(f)$.

where $W(f) = \int_{\mathcal{U}(F)\backslash\mathcal{U}(\mathbb{A})} \psi(u) f(u a_0)\, du$ Whit period func'n

$f: \mathrm{Bun}_G(\mathbb{F}_q) \to k$, $\psi: \mathcal{U} \to \mathbb{G}_a$, $a_0 = e^{2\check{\rho}}(\varpi^{-1/2})$.

Have $\|\mathcal{P}_X^{\mathrm{norm}}\|_{L^2} \approx 1$.

<u>Rmk</u> (i) Can generalize to $M = T^*(X, \Psi)$, $X = S^+ \times^{H\mathcal{U}} G$,

Ψ = an affine $\mathbb{G}_a$-bundle over X

(e.g. $\Psi = S^+ \times^{H\mathcal{U}'} G$ where $\mathcal{U}' = \ker(\psi: \mathcal{U} \to \mathbb{G}_a)$).

(2) This is as predicted by Lapid–Mao.
(Suffices to look at the power of q.)

§3 Whittaker induction and Eisenstein series

Eisenstein setting:

$$(G,M) = (SL_2, T^*\mathbb{A}^2) \longleftrightarrow (PGL_2, T^*(\mathbb{G}_m \backslash PGL_2)) = (\check{G}, \check{M})$$

A general model of $G = SL_2 \circlearrowright X = \mathbb{A}^2 - \{0\}$ $\quad (g \mapsto (0,1)g)$:

$X = U\backslash G \circlearrowleft G \times T$ via $(g,t) : Ux \longmapsto Ut^{-1}xg$

$\circlearrowleft G \times \mathbb{G}_{gr}$ via $(1, e^{-2\rho})$: $\lambda \in \mathbb{G}_{gr}$ acts on X by $\lambda^{2\rho} \in G$.

Toroidal-type compactification

<u>Fact</u> $\Psi \to X$ as before $\rightsquigarrow$ $X \to B\mathbb{G}_a$ $\quad (G \times \mathbb{G}_{gr})$-equivariant.

Also, $G \circlearrowright \mathbb{G}_a$ trivially $\rightsquigarrow$ $X/G \to B\mathbb{G}_a = pt/\mathbb{G}_a$

$\mathbb{G}_{gr} \circlearrowright \mathbb{G}_a$ by square char $\rightsquigarrow$ $X/(G \times \mathbb{G}_{gr}) \to pt/(\mathbb{G}_a \rtimes \mathbb{G}_{gr})$.

So we get $\quad \mathrm{Map}(\Sigma, X/(G\times\mathbb{G}_{gr})) \longrightarrow \mathrm{Map}(\Sigma, pt/(\mathbb{G}_a \rtimes \mathbb{G}_{gr}))$

$$\begin{aligned}
\rightsquigarrow \quad & \mathrm{Map}(\Sigma, X/G) = \mathrm{Bun}_G^X(\Sigma) = \text{fiber of } \mathrm{Map}(\Sigma, X/(G\times\mathbb{G}_{gr})) \text{ on } \mathcal{K}^{1/2} \\
& \qquad \Big\downarrow \text{natural} \\
& \mathrm{Map}(\Sigma, X/(G\times\mathbb{G}_{gr})) \\
& \qquad \Big\downarrow \\
& \mathrm{Map}^{\mathcal{K}^{1/2}}(\Sigma, pt/(\mathbb{G}_a\rtimes\mathbb{G}_{gr})) := \left\{ \mathcal{K}_\Sigma^{-1/2}\setminus\{0\} \xrightarrow{\mathbb{G}_m\text{-equiv map}} pt/(\mathbb{G}_a\rtimes\mathbb{G}_{gr}) \right\} \\
& \qquad \Big\downarrow \wr \\
& \underbrace{H^1(\Sigma,\Omega^1)}_{\mathbb{G}_a = \mathbb{A}^1 \text{ as grp sch}} / H^0(\Sigma,\Omega^1) \longrightarrow \mathbb{G}_a
\end{aligned}$$

Back to $G = SL_2$, $X = \mathbb{G}_m^2 = \mathbb{A}^2 - \{0\}$.

Along $\mathrm{Bun}_G^X \xrightarrow{\pi} \mathrm{Bun}_G$

$$\begin{aligned} &\quad\quad\quad\quad\quad\quad\quad \mathcal{E} \in \mathrm{Bun}_G \text{ rank 2 unimodular vec bundle} \\ \pi^{-1}(\mathcal{E}) &= \Gamma(\Sigma, \mathcal{E}\otimes \mathcal{K}^{1/2}) \\ &= \{\text{everywhere injective maps } \mathcal{K}^{1/2} \to \mathcal{E}\} \\ &= \{\mathcal{K}^{1/2} \to \mathcal{E} \to \mathcal{K}^{-1/2}\} \end{aligned}$$

$$\Rightarrow \mathrm{Bun}_G^X(\Sigma) = \underbrace{H^1(\Sigma, \mathcal{K})}_{\simeq \mathbb{A}^1} / \underbrace{H^0(\Sigma, \mathcal{K})}_{\text{unipotent grp sch}}$$

More generally, $X = U \backslash G \circlearrowleft \mathbb{G}_{gr}$ nontrivial

$$\Rightarrow \mathrm{Bun}_G^X(\Sigma) = \mathbb{A}^r/\mathcal{U}, \quad r = \dim(\text{span of simple roots of } G).$$

$$\begin{array}{ccc} \mathrm{Bun}_G^X(\Sigma) & & \mathbb{G}_m^r \\ \downarrow \pi & & \downarrow \\ \mathrm{Bun}_G(\Sigma) & \ni & pt. \end{array}$$

Rmk π is not a closed immersion but it factors as

$$\begin{tikzcd} \mathrm{Bun}_G^X(\Sigma) \arrow[r, "\pi"] \arrow[d] & \mathrm{Bun}_G(\Sigma) \\ \mathrm{Bun}_G^X(\Sigma)/T \arrow[ur, hook, "\text{locally closed immersion.}"'] & \end{tikzcd}$$

Whittaker induction (sheaf ver)

Fix $H \times SL_2 \to G$, $\psi: U \to \mathbb{G}_a$ with $U \subset G$ unip subgrp.

- Structure thm of Hamiltonian G-spaces
  $\rightsquigarrow$ $w\text{-}\mathrm{ind}_H^G$: {graded Hamil H-space} $\to$ {graded Hamil G-spaces}.
- Upgrade ψ to $\psi: \mathrm{Bun}_U^{\mathcal{K}^{1/2}}(\Sigma) := \mathrm{Map}^{\mathcal{K}^{1/2}}(\Sigma, pt/U) \to \mathbb{G}_a$.
- Have Artin-Schreier sheaf $\mathcal{L} \in \mathrm{Shv}(\mathbb{G}_a)$.

Take Fourier transform

$$WI_H^G : Shv(Bun_H) \longrightarrow Shv(Bun_G)$$
$$\mathcal{F} \longmapsto \pi_{2!}(\pi_1^*\mathcal{F} \otimes \psi^*\mathcal{L})$$

$$\begin{tikzcd} & Bun_{HU}^{K^{1/2}}(\Sigma) \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \\ Bun_H(\Sigma) & & Bun_G(\Sigma) \end{tikzcd}$$

This gives rise to a functor

$$Shv(Bun_{HU}^{K^{1/2}}) \longrightarrow Hom(Shv(Bun_H), Shv(Bun_G)) .$$

<u>Lem 10.8.2</u> Suppose $w\text{-}ind_H^G(T^*Y) = T^*X$.

Then $WI_H^G(\mathcal{P}_Y) \simeq \mathcal{P}_X$

where $\mathcal{P}_Y$ = period sheaf on Bun_H.

(Compatible w/ Thm 3.6.1).

<u>Rmk</u> Have a corresponding phenomenon on spec side (11.9.2).

<u>Punchlines</u> (1) This reduces everything to symplectic case.

(2) Fixing a symplectic "base period $\mathcal{P}_Y$",
can describe $\mathcal{P}_X$ by a functor $Shv(Bun_H) \to Shv(Bun_G)$.

<u>Eisenstein series</u>

Assume $G_{gr} \curvearrowright X = U\backslash G$ trivially. Then

$$\begin{array}{ccc} G\times T \curvearrowright X = U\backslash G & \longleftrightarrow & \check{X} = \check{U}\backslash\check{G} \curvearrowleft \check{G}\times\check{T} \\ \text{\rotatebox{90}{$\cong$}} & & \text{\rotatebox{90}{$\cong$}} \\ B\backslash(G\times T) & & (\check{G}\times\check{T})/\check{B}^- \end{array}$$

<u>Peculiarity</u> (a) M, $\check{M}$ not affine

$\Rightarrow$ (G, M), $(\check{G}, \check{M})$ not Hamil dual pairs

$\Rightarrow$ the formulae for $\mathcal{P}_X$ & P_X may fail to be valid

(b) It has an interaction w/ GLC.

$\rightsquigarrow$ can read info through the functor.

<u>Rough idea to remedy</u>

$$\begin{tikzcd}[column sep=small] & \mathrm{Bun}_B \arrow[dl, "q"'] \arrow[dr, "p"] & & & \mathrm{Loc}_{\check{B}} \arrow[dl, "\check{q}"'] \arrow[dr, "\check{p}"] & \\ \mathrm{Bun}_T & & \mathrm{Bun}_G & \mathrm{Loc}_{\check{T}} & & \mathrm{Loc}_{\check{G}} \end{tikzcd}$$

$\rightsquigarrow$ $\mathrm{Eis}_{\mathrm{spec}} := \check{p}_* \check{q}^! : \mathrm{QC}^!(\mathrm{Loc}_{\check{T}}) \to \mathrm{QC}^!(\mathrm{Loc}_{\check{G}})$

$\mathrm{Eis}_! := p_! q^* : \mathrm{SHV}(\mathrm{Bun}_T) \to \mathrm{SHV}(\mathrm{Bun}_G)$.

Can rewrite $\mathcal{P}_x \longleftrightarrow \mathcal{L}_{\check{x}}$ as $\mathrm{Eis}_! \longleftrightarrow \mathrm{Eis}_{\mathrm{spec}}$.

<u>Conj 12.3.4</u> (Strange functional equation)

$$\begin{tikzcd} \mathrm{SHV}(\mathrm{Bun}_T) \arrow[r, "\mathbb{L}_T"] \arrow[d, "\mathcal{P}_x^{\mathrm{spec}}"'] \arrow[dr, phantom, "\circlearrowright"] & \mathrm{QC}^!(\mathrm{Loc}_{\check{T}}) \arrow[d, "\mathcal{L}_{\check{x}}^{d}"] \\ \mathrm{SHV}(\mathrm{Bun}_G) \arrow[r, "\mathbb{L}_G"] & \mathrm{QC}^!(\mathrm{Loc}_{\check{G}}) \end{tikzcd}$$

dualizing involution

"$\mathrm{Eis}_!$" "$\mathrm{Eis}_{\mathrm{spec}}$"

to be discussed in §12.