L-sheaves (II)

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<u>Today's goal</u>
- L-sheaves & L-fcts
- Examples
- Functional equation
  (independence of polarization).

<u>Slogan</u> L-sheaves $\rightsquigarrow$ special values of L-fcts
$\mathbb{G}_{gr}$ $\rightsquigarrow$ points to evaluate at

<u>Recall</u> Σ curve $/\mathbb{F}_q$, $k = \overline{\mathbb{Q}}_\ell$, $H^*(-) := H^*_{geom}(\Sigma_{\overline{\mathbb{F}}_q}, -)$.

T loc sys $\rightsquigarrow$ $L(s,T) := \prod \det(1 - q^{-s}\mathrm{Fr} \mid H^i(T))^{(-1)^{i+1}}$

$$\varepsilon_{\frac{1}{2}}(s,T) := \det(T)\cdot(\partial^{1/2})\cdot q^{(1/2 - s)(g-1)\dim T}.$$

$$L^{norm}(s,T) := L(s,T)\cdot \varepsilon_{\frac{1}{2}}^{-1}(s,T)$$

$$\pi_1 \xrightarrow{\rho} \check{G} \times \mathbb{G}_{gr} \to GL(T)$$

$$\pi_1 \to F^\times \backslash \mathbb{A}_F^\times / \prod \mathcal{O}_{F_v}^\times \xrightarrow{|\cdot|} \mathbb{G}_{gr}$$

$$\rightsquigarrow L(s, T^{\#}) := \prod_k L(s + \tfrac{k}{2}, T_k)$$

Similarly, define $\varepsilon_{\frac{1}{2}}(s, T^{\#})$ & $L^{norm}(s, T^{\#})$.

$\Omega^{\frac{1}{2}} = \mathcal{O}(\Sigma \frac{n_v}{2}\cdot v)$.

<u>L-sheaf</u> $\mathcal{L}_{\check{X}} := (\pi_* \omega_{\mathrm{Loc}_{\check{G}}^{\check{X}}})^{\#} \in QC^!(\mathrm{Loc}_{\check{G}})$

for

$$\begin{array}{ccc} \pi : \mathrm{Loc}_{\check{G}}^{\check{X}} & \longrightarrow & \mathrm{Loc}_{\check{G}} \\ \| & & \| \\ \mathrm{Map}(\Sigma_{et}, \check{X}/\check{G}) & & \mathrm{Map}(\Sigma_{Betti/dR}, B\check{G}). \end{array}$$

$$\begin{tikzcd}[column sep=small]
\text{Take} \quad \mathcal{E}_{\frac{1}{2}} = [\Omega^{1/2}] & \in & \mathrm{Pic}(\mathrm{Loc}_{\mathbb{G}_m}) & \ni & {[\mathcal{O}(x)]_1 = L_x} \\
\Omega^{1/2} \arrow[u, mapsto] & \in & \mathrm{Pic}(\Sigma) \arrow[u] & \ni & \mathcal{O}(x) \arrow[u, mapsto]
\end{tikzcd}$$

& $[\mathcal{E}_{\frac{1}{2}}(T)] = \mathcal{E}_{\frac{1}{2}}(\tfrac{1}{2}, T)$.

<u>Def</u> $\mathcal{L}_{\check{X}}^{norm} := \mathcal{L}_{\check{X}} \otimes \check{\eta}^* \check{\mathcal{E}}_{\frac{1}{2}} \langle -\beta_{\check{X}} \rangle$.

where $\beta_{\check{X}} := (g-1)(\dim G + \gamma - \dim X)$

$\mathbb{G}_{gr} \times \check{G} \circlearrowright \check{X}$, $\exists\, \omega \in H^0(\check{X}, \Omega^{top})$,

$$(\lambda, g)^* \omega = \check{\eta}(g) \lambda^{\gamma} \cdot \omega$$

with $\check{\eta}: \check{G} \to \mathbb{G}_m$, $\gamma \in \mathbb{Z}$.

<u>Topologize</u> $\rho: \pi_1^{ét}(\Sigma) \longrightarrow \check{G}(k)$

$$\rightsquigarrow \quad \begin{tikzcd}
L_\rho : * \arrow[rr] \arrow[dr] & & \mathrm{Loc}_{\check{G}} \arrow[dl, dashed] \\
& {[*/Z(\rho)]} &
\end{tikzcd}$$

$\delta_\rho = \varphi_* k \in QC^!(\mathrm{Loc}_{\check{G}})$

$Z(\rho) := Z_{\check{G}}(\rho^{geom})$,

$d = \dim Z(\rho)$

<u>Theorem</u> Conditionally,

$$(*) \quad \mathrm{Hom}(\delta_\rho, \mathcal{L}_{\check{X}}) = (\mathrm{Sym}^* H^* T^{\square})[-d]^{\wedge}$$

$$\mathrm{Hom}(\delta_\rho, \mathcal{L}_{\check{X}}^{norm}) = (\mathrm{Sym}^* H^* T^{\square})[-d]^{\wedge} \otimes \check{\mathcal{E}}_{\frac{1}{2}}(T) \langle -\beta_{\check{X}} \rangle$$

$$(**) \quad \mathrm{Hom}(\mathcal{L}_{\check{X}}, \delta_\rho) = (\mathrm{Sym}^* H^* T^{\square})^{\vee} \otimes \det H^1(\mathrm{ad}\,\rho)^{\wedge}$$

$$\mathrm{Hom}(\mathcal{L}_{\check{X}}^{norm}, \delta_\rho) = (\mathrm{Sym}^* H^* T^{\square})^{\vee} \otimes \det H^1(\mathrm{ad}\,\rho)^{\wedge} \otimes \mathcal{E}_{\frac{1}{2}}(T) \langle \beta_{\check{X}} \rangle$$

$\underset{\rightsquigarrow}{\mathrm{Tr(Frob)}}$

$$(-1)^d L(1, T^{\square\vee})$$

$$(-1)^d \cdot q^{-b_{\check{G}}/2} L^{norm}(1, T^{\square\vee})$$

$$q^{-b_{\check{G}}/2} L(0, T^{\square})$$

$$q^{-b_{\check{G}}/2} L^{norm}(0, T^{\square}).$$

<u>Completion</u> $T = \mathrm{Rep}(Z(\rho)) \longrightarrow \hat{T} := \mathrm{Hom}_{Z(\rho)}(k[Z(\rho)], T)$

$$\begin{array}{ccc} \| & & \| \\ \bigoplus T_{\alpha'} & \longrightarrow & \prod T_\alpha \end{array} \quad \text{for } \alpha > 0.$$

(*) untwisted, polarized, $\pi_1^{geom} \xrightarrow{\rho} \check{G} \circlearrowright \check{X}$,

$\check{X}^{\rho^{geom}} = \{x_0\}$ reduced singleton

$\Rightarrow H^0(\Sigma, T) = H^2(\Sigma, T) = 0$.

$\uparrow \mathrm{Hom}(\mathrm{Spec}\, k[\varepsilon]/\varepsilon^2, \check{X})^{\pi_1} = 0$

$\rightsquigarrow \mathrm{Spec}\, k[\varepsilon]/\varepsilon^2 \to \check{X}^\rho$.

(**) (*) + ρ is sm pt of $\mathrm{Loc}_{\check{G}}$ ($\Leftrightarrow H^0(\mathrm{ad}\,\rho) = H^2(\mathrm{ad}\,\rho) = 0$).

+ $i_\rho : BZ(\rho) \to \mathrm{Loc}_{\check{G}}$ eventually coconnective

($\Rightarrow$ so is $i_\rho^* : \mathrm{Coh}(X) \to \mathrm{Coh}(X)$)

$\Rightarrow i_\rho^* = i_\rho^! \langle \cdots \rangle$.

<u>Examples</u> (1) Whittaker case

$X = G/U \longleftrightarrow \check{X} = pt, \ \mathcal{L}_{\check{X}} = \omega$

const stalks $\longleftrightarrow T = 0 \rightsquigarrow$ triv L-fct.

(2) Iwasawa-Tate case

$\rho = \chi : \pi_1 \to k^\times, \ Z(\rho) = \mathbb{G}_m \rightsquigarrow T = \chi \times^{\mathbb{G}_m} \mathbb{A}^1$

$\mathbb{G}_{gr} \circlearrowright \mathbb{A}^1$ by scaling.

$$\begin{tikzcd}
\mathrm{Loc}^{\mathbb{A}^1}_{\mathbb{G}_m} \arrow[d, "\pi"'] & R\Gamma(\Sigma, T)/\mathbb{G}_m \arrow[l] \arrow[d] & R\Gamma(\Sigma, T) \arrow[l] \arrow[d] \rightsquigarrow \mathrm{Spec}\,\mathrm{Sym}\, H^1(\Sigma, T)[-1]) \\
\mathrm{Loc}_{\mathbb{G}_m} & B\mathbb{G}_m \arrow[l, "i_\rho"'] & * \arrow[l, "p"']
\end{tikzcd}$$

$$\mathrm{Hom}_{\mathrm{Loc}_{\mathbb{G}_m}}(i_{\rho *} p_* k, \pi_* \omega^{\square}) = \mathrm{Hom}_{\mathrm{Rep}\,\mathbb{G}_m}(k[\mathbb{G}_m], \pi_* \omega^{\square}_{R\Gamma(\Sigma,T)}[-1])$$

$$= \Big(\big(\underbrace{(\pi_* \mathcal{O}_{R\Gamma(\Sigma,T)})}_{\mathrm{Sym}\, H^1(\Sigma,T)[-2]}{}^{\vee} \big)^{\square} \Big)^{\wedge} [-1] =$$

<u>Note</u> $\mathrm{Tr}(\mathrm{Frob}, \mathrm{Sym}\, H^1(\Sigma, T)[-1]) = \det(1-\mathrm{Frob}, H^1(\Sigma, T))$.

(3) Hecke case:

$GL_2 \circlearrowright GL_2/\begin{pmatrix}1 & \\ & *\end{pmatrix} \longleftrightarrow \mathbb{A}^2 \circlearrowleft GL_2$

$\rho : \pi_1 \longrightarrow GL_2$ irred (like $\begin{pmatrix}\chi_1 & * \\ & \chi_2\end{pmatrix}$).

$Z(\rho) = \mathbb{G}_m \rightsquigarrow L(-, \rho)$.

<u>pf sketch of thm</u>

<u>Check normalization</u> $\sum \dim T_k = \dim T = \dim \check{X}$, $\sum k \cdot \dim T_k = \gamma$.

Fr trace of ε: $\varepsilon(1, T^{\vee \square}) = \prod \varepsilon(1+\frac{k}{2}, T_{-k}^{\vee})$

$= \prod \varepsilon(\frac{1}{2}, T_{-k}^{\vee}) \cdot q^{\frac{-1-k}{2} \cdot (2g-2) \dim T_{-k}}$

$= \varepsilon(\frac{1}{2}, T^{\vee}) \cdot q^{(g-1)\sum(k-1)T_k}$ $= \gamma - \dim \check{X}$.

$\Rightarrow [(\mathcal{E}_{\frac{1}{2}}^{\vee}(T)\langle -\beta_{\check{X}} \rangle)^{\vee}] = q^{-\beta_{\check{X}}/2} \cdot \varepsilon(\frac{1}{2}, T^{\vee})^{-1/2}$

$= q^{\frac{1}{2}((g-1)(\gamma - \dim \check{X}) - \beta_{\check{X}})} \varepsilon(1, T^{\square\vee})^{-1/2}$

$= q^{-b_{\check{X}}/2} \cdot \varepsilon_{\frac{1}{2}}^{-1}(1, T^{\square,\vee})$.

<u>sheaf calculation</u>

$$\begin{tikzcd}
\mathrm{Loc}_{\check{G}}^{\check{X}} \arrow[d, "\pi"'] & {[\mathrm{Loc}_{\rho}^{\check{X}}/Z(\rho)]} \arrow[l] \arrow[d, "\pi"] & \mathrm{Loc}_{\rho}^{\check{X}} \arrow[l] \arrow[d, "\pi"] \\
\mathrm{Loc}_{\check{G}} & {[*/Z(\rho)]} \arrow[l, hook', "i_\rho"'] & * \arrow[l, "\rho"']
\end{tikzcd}$$

$\mathrm{Loc}_{\rho}^{\check{X}} = \mathrm{Map}_{B\check{G}}(\Sigma_{\text{ét}}, \check{X}/\check{G}) = R\Gamma(\Sigma, T)$

$(H^0(\Sigma, T) = H^2(\Sigma, T) = 0)$

closed imm $\Rightarrow$ proper.

lci + eventually coconn.

- $\mathrm{Loc}_{\rho}^{\check{X}} = \mathrm{Map}_{B\check{G}}(\Sigma_{\text{ét}}, \check{X}/\check{G}) = \mathrm{Map}_{B\check{G}}(B\pi_1, \check{X}/\check{G})$

$= \mathrm{Map}_{B\pi_1}(B\pi_1, \check{X}/\pi_1) = \mathrm{Map}_{\pi_1}(*, \check{X})$

$= (\check{X})^{\pi_1} \approx (T)^{\pi_1}$

- $\mathrm{Hom}(S_\rho, \mathcal{L}_{\check{X}}) = \mathrm{Hom}(i_{\rho *}\rho_* k, \pi_* \omega^{\square})$

$$\mathrm{QCoh}(BZ(\rho)) \xrightarrow[\sim]{p^*} \mathrm{Rep}\, Z(\rho) \qquad \begin{aligned} &= \mathrm{Hom}_{BZ(\rho)}(p_* k, \pi_* \omega^{\square}_{[\mathrm{Loc}^{\check{X}}_{\check{G}}/Z(\rho)]}) \\ &= \mathrm{Hom}_{\mathrm{Rep}\, Z(\rho)}(k[Z(\rho)], \pi_* \omega^{\square}_{R\Gamma(T)})[-d] \\ &= \left(\left(\left(\pi_* \mathcal{O}_{R\Gamma(\Sigma,T)}\right)^{\vee}\right)^{\square}\right)^{\wedge}[-d] \end{aligned}$$

$$p^* = p^![-d] \qquad\qquad \mathrm{Sym}\, H^1(\Sigma, T)^{\vee}[-1] \text{ (is)} .$$

- $\mathrm{Hom}(\mathcal{L}_{\check{X}}, \delta_\rho) = \mathrm{Hom}_{\mathrm{Rep}\, Z(\rho)}(p^! \pi_* \zeta_\rho^* \omega^{\square}, p^! p_* k)$
$$= (\mathrm{Sym}\, H^1(T)[-1]^{\square})^{\vee\wedge} \otimes \det H^1(\mathrm{ad}\,\rho)\, [d - \dim T_\rho]$$
$$\dim T_\rho = 2b_{\check{G}} .$$

§ <u>Functional equation</u>

$$L^{\mathrm{norm}}(s, T) = L^{\mathrm{norm}}(1-s, \check{T}).$$

<u>Safety ass'n</u> $\check{X}$ vectorial, Betti,

$\check{G}$ ss, $g = \mathrm{genus}(X) \geq 2$,

$QC^! \longrightarrow QC$ spectral projection.

<u>Thm</u> $\mathcal{L}_{\check{X}} \simeq \mathcal{L}_{\check{X}^\vee} \otimes \underbrace{\det H^*(\check{X})^{\otimes -1}}_{\in \mathrm{Pic}(\mathrm{Loc}_{\check{G}})}$.

$\mathcal{L}^{\mathrm{norm}}_{\check{X}} \simeq \mathcal{L}^{\mathrm{norm}}_{\check{X}^\vee}$.

pf. Check $\beta_{\check{X}} = \beta_{\check{X}^\vee} = (g-1)(\dim \check{G} + \gamma - \dim X)$

$(\gamma = \dim \check{X}$ if $\check{X}$ vectorial$)$.

$$\check{\eta}^* \check{\mathcal{E}}_{\frac{1}{2}} \otimes (\check{\eta}) \check{\mathcal{E}}_{\frac{1}{2}} = (\check{\eta}^* \mathcal{E})^{-1}$$

<u>Notice</u> $\check{\eta}^* \mathcal{E}_{\frac{1}{2}}$ = square root of $\det H^*(\check{X})$.

$$\begin{aligned} [H^*(\Sigma, T)] &= [H^*(\Sigma, T \otimes \mathcal{O} \to T \otimes \Omega^1)] \\ &= [H^*(\Sigma, T \otimes \mathcal{O})] - [H^*(\Sigma, T \otimes \Omega^1)] \\ &= \sum n_v \cdot [T_v]. \end{aligned}$$

$$\begin{array}{ccc} \mathrm{Loc}^{\check{X}}_{\check{G}} = R\Gamma(\Sigma, \check{X} \times^{\check{G}} \mathcal{E}^{\mathrm{univ}}) & \longleftarrow & \underline{E} = [\overset{0}{P} \to \overset{1}{Q} \to \overset{2}{R}] \text{ (perf complex)} \\ \downarrow & & \downarrow \\ \mathrm{Loc}_{\check{G}} = \mathrm{Spec}\, A / \check{G} & \longleftarrow & \mathrm{Spec}\, A := \{(a_i, b_i) \in \check{G}^2,\ 1 \leq i \leq g \mid \prod [a_i, b_i] = 1\}. \end{array}$$

$\underline{\text{Fact}}$ A non-derived & lci

$$\Rightarrow \mathbb{L}_A \in \mathrm{Perf}(A)^{[-1,0]}, \quad \omega_A \in \mathrm{Pic}(D(A)).$$

Compute $E \xrightarrow[\text{proper}]{\pi_0} P \xrightarrow[\text{smooth}]{p} \operatorname{Spec} A$.

$$\rightsquigarrow \mathcal{L}_{\check{X}}|^!_{\operatorname{Spec} A} = \mathrm{Hom}(\mathcal{O}, \pi_0^! \omega_P)^{\square}$$

$$= \mathrm{Hom}_{\mathcal{O}_P}(\pi_{0*}\mathcal{O}, \omega_A \otimes \det P^\vee [\dim P] \otimes \mathcal{O}_P)^{\square}$$

$$= \mathrm{Hom}_{\mathrm{Sym}\, P^\vee[-1]}(\mathrm{Sym}\, \check{E}[-1], \mathrm{Sym}\, \check{P}[-1]) \otimes \omega_A \otimes \det \check{P}.$$

$$= \Lambda^* Q \otimes \mathrm{Sym}((R \oplus P)^\vee[-1]) \otimes \det \check{P} \otimes \omega_A.$$

Dually, $\mathcal{L}_{\check{\check{X}}}|^!_{\operatorname{Spec} A} = \underbrace{\Lambda^* Q^\vee}_{= \Lambda^* Q \otimes \det Q^{-1}} \otimes \mathrm{Sym}((R \oplus P^\vee)[-1]) \otimes \det R \otimes \omega_A$

Their difference $= \det P \otimes \det Q^\vee \otimes \det R$

$$= \det H^*(\check{X}).$$

For $(\Lambda^* Q \otimes \mathrm{Sym}(R \oplus P^\vee[-1]), d)$,

Choose bases $\{x_1, \dots, x_n\}$ of Q & $\{x_1^*, \dots, x_n^*\}$ of $Q^\vee$.

$$x_j \longmapsto \hat{x}_j\, d x_j \in R, \quad \hat{x}_j \longmapsto x_j\, d^\vee x_j^\vee \in P^\vee.$$

- differential is $\mathrm{Sym}(R \oplus P^\vee[-1])$
- $d(x_1 \wedge \dots \wedge x_n) = \sum_{1 \le j \le n} (-1)^{j-1}.$ $\square$