Global geometric Conjecture (I)

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Σ proj sm curve, geom irred $/\mathbb{F}$

k coeff field of char 0.

<u>Rough form of the conj</u>

Assume $(G, M = T^*(X, \Psi))/\mathbb{F} \longleftrightarrow (\check{G}, \check{M} = T^*(\check{X}, \check{\Psi}))/k$

(split form)

Under the geom Langlands corr,

$$\begin{array}{ccc} AUT(Bun_G(\Sigma)) & \simeq & QC^!(Loc_{\check{G}}(\Sigma)) \\ \cup & & \cup \\ \mathcal{P} = \mathcal{P}_X^{norm} & \longleftrightarrow & \mathcal{L} = \mathcal{L}_X^{norm} \end{array}$$

<u>Global geom Langlands corr</u> (statement)

$$AUT_{①}^{②}(Bun_G(\Sigma)) \simeq QC_{①}^!(Loc_{\check{G}}^{②}(\Sigma))$$

② $\in \{dR, B, et\}$, ① something else.

<u>Subtlety of topological sheaves on stacks</u> (Appendix B)

For X stack, define <u>safe cat</u> of shv

$$SHV_s(X) := \lim_{\substack{f: Y \to X \\ Y \text{ affine fin type}}} (\underline{SHV}(Y), f^!).$$

$$Shv_s(X) := \lim_{\substack{R \text{ fin type} \\ f: \operatorname{Spec} R \to X}} (Shv^{constr}(Y), f^!).$$

<u>Issue</u> For schemes Y of fin type,

$\mathrm{Shv}^{\mathrm{constr}}(Y) = \mathrm{SHV}(Y)^{\omega}$ compact objs

But for general stacks this is not true.

$\mathrm{Shv}(X)$ need not be cpt objs in $\mathrm{SHV}_s(X)$.

e.g. $X = \mathbb{B}G$, $\mathrm{SHV}_S(X) = H_*(G)\text{-mod}$

$\underset{\text{triv } G\text{-act}}{k}$ = augmentation mod $\in \mathrm{Shv}(X)$.

But it is not quasi-isom to a perfect complex of $H_*(G)$-mod.

<u>Solution</u> $\mathrm{Shv}_s(X) := \mathrm{SHV}_s(X)^{\omega} \subseteq \mathrm{Shv}(X)$

Define $\mathrm{SHV}(X) := \mathrm{Ind}(\mathrm{Shv}(X))$.

$$\rightsquigarrow \begin{tikzcd} \mathrm{Shv}_s(X) \arrow[r, hook, "\omega"] \arrow[d, hook] & \mathrm{SHV}_s(X) = \mathrm{Ind}(\mathrm{Shv}_s(X)) \arrow[d, hook, "\cap"] \\ \mathrm{Shv}(X) \arrow[r, hook, "\omega"] \arrow[ur, "\text{natural}"] & \mathrm{SHV}(X) = \mathrm{Ind}(\mathrm{Shv}(X)). \end{tikzcd}$$

<u>Rmk</u> The natural map $\mathrm{Shv}(X) \to \mathrm{SHV}_s(X)$ induces a functor

$\mathrm{safe}: \mathrm{SHV}(X) \longrightarrow \mathrm{SHV}_s(X)$

"colocalization".

* <u>de Rham setup</u>

$$\mathrm{AUT}^{dR}(\mathrm{Bun}_G(\Sigma)) := \mathrm{SHV}^{dR}(\mathrm{Bun}_G(\Sigma)) = \mathfrak{D}(\mathrm{Bun}_G(\Sigma))$$
$$= \mathrm{Ind}(\underbrace{\mathrm{Shv}^{dR}(\mathrm{Bun}_G(\Sigma))}_{\text{holonomic objs} \subseteq \text{coherent D-mods}})$$

* <u>Betti setup</u>

$$\mathrm{AUT}^B(\mathrm{Bun}_G(\Sigma)) := \mathrm{SHV}^B_{\mathcal{N}}(\mathrm{Bun}_G(\Sigma)) \hookrightarrow \mathrm{SHV}^B(\mathrm{Bun}_G(\Sigma))$$

Singular supp $\subseteq \mathcal{N}$ = nilp cone

of $T^*(\mathrm{Bun}_G(\Sigma)) = \mathrm{Map}(\pi_1^B(\Sigma), \mathfrak{g}^*[-1]/G)$

$$\cup\!\!\!\mid$$

$$\mathrm{Map}(\pi_1^B(\Sigma), \mathcal{N}^*[-1]/G).$$

$\mathrm{SHV}^B_{\mathcal{N}}$ = "largest subcat" admitting loc const Hecke action.

$\exists$ a "left" spectral projection

$$(-)^{spec}: \mathrm{SHV}^B \longrightarrow \mathrm{SHV}^B_{\mathcal{N}}$$
$$M \longmapsto M^{spec}$$

* <u>étale setup</u>

$\mathrm{AUT}^{et}(\mathrm{Bun}_G(\Sigma)) := \mathrm{SHV}^{et}_{\mathcal{N}}(\mathrm{Bun}_G(\Sigma))$ (c.f. §12.4).

Same as above, expect $M \mapsto M^{spec}$ is the "right" projector.

<u>Rmk</u> $\mathrm{AUT}^{②}_{s}(\mathrm{Bun}_G(\Sigma))$ w/ ind-safe condition.

<u>Rmk</u> $\underbrace{\mathrm{SHV}^B_{\mathcal{N}}(\mathrm{Bun}_G(\Sigma))}$ is quite different from étale ver $\mathrm{SHV}^{et}_{\mathcal{N}}(\mathrm{Bun}_G(\Sigma))$.

cpt objs need not have
finite-rank cohom shvs.

e.g. $\mathrm{SHV}^B_{\mathcal{N}}(\mathbb{G}_m) = k[\pi_1(\mathbb{G}_m)]$-mod.

- cpt objs are pres. $k[\pi_1(\mathbb{G}_m)]$-mod
- <u>But</u> need not be fin dim'l $/k$.

<u>Spectral side</u>

de Rham: $\mathrm{Loc}^{dR}_{\check{G}}(\Sigma)$:= moduli of flat $\check{G}$-connections on Σ

Betti: $\mathrm{Loc}^{B}_{\check{G}}(\Sigma)$:= moduli of loc const $\check{G}$-torsors on Σ.

étale: $\mathrm{Loc}^{\text{ét}}_{\check{G}}(\Sigma) :=$ moduli of $\otimes$-functors $[\mathrm{Rep}(\check{G}) \to \{\text{cont reps of } \pi_1^{\text{ét}}(\Sigma)\}]$.

When $\mathbb{F} = \mathbb{C}$:

$$\mathrm{Loc}^{\mathrm{dR}}_{\check{G}}(\Sigma) \longleftarrow \mathrm{Loc}^{\text{ét}}_{\check{G}}(\Sigma) \hookrightarrow \mathrm{Loc}^{B}_{\check{G}}(\Sigma)$$

same as formal completion at semisimple rep'ns.

<u>Note</u> For a dg stack X,

$$\begin{array}{ccccc} \mathrm{Perf}(X) & \subseteq & \mathrm{Ind}(\mathrm{Perf}(X)) = \mathrm{QC}(X) & \xleftarrow{\Psi_X} & \\ & & \cup\!\!\!\mathsf{I} & & \\ & & \mathrm{Coh}(X) \subseteq \mathrm{Ind}(\mathrm{Coh}(X)) = \mathrm{QC}^!(X) & & \end{array}$$

$\mathrm{Coh}(X)$ = Complex of shvs of X w/ fin coh cohom.

- Ψ_X is essentially surj if $\mathcal{O}_X$ is supp on finitely many degs.
- $\mathrm{Perf}(X) \hookrightarrow \mathrm{Coh}(X)$

  Take Ind, $\Xi : \mathrm{QC}(X) \longrightarrow \mathrm{QC}^!(X)$.

<u>Caveat</u>: $\mathrm{Coh}(X) \subseteq \mathrm{QC}(X) \xrightarrow{\Xi} \mathrm{QC}^!(X)$

is <u>NOT</u> the same as $\mathrm{Coh}(X) \hookrightarrow \mathrm{QC}^!(X)$.

<u>E.g.</u> $X = \mathrm{Spec}\,\Lambda$, $\Lambda = k[x]/(x^2)$. $y_{-1}^2 = 0$.

issue: $\longrightarrow \Lambda \xrightarrow{\cdot x} \Lambda \xrightarrow{\cdot x} \Lambda \xrightarrow{\cdot x} \cdots$

is trivial in $\mathrm{QC}(X)$ but not in $\mathrm{QC}^!(X)$.

① $k \in \mathrm{Coh}(X)$ can be viewed as

$$\cdots \to 0 \to 0 \to \Lambda \xrightarrow{\cdot x} \Lambda \xrightarrow{\cdot x} \cdots \xrightarrow{\cdot x} \Lambda \to \cdots$$

② $k \in \mathrm{Ind}(\mathrm{Perf})$ can be viewed as

$$\cdots \to \Lambda \xrightarrow{\cdot x} \cdots \xrightarrow{\cdot x} \Lambda \xrightarrow{\cdot x} \Lambda \to 0 \to 0.$$

$\Xi(k)$ essentially the same as k.

Global geom Langlands corr (statement)

$$
\begin{tikzcd}
AUT_s^{\circledast}(Bun_G(\Sigma)) \arrow[r, phantom, "\simeq"] \arrow[d] & QC^!_{\mathcal{N}}(Loc^{\circledast}_{\check{G}}(\Sigma)) \circlearrowleft QC^{\text{II}}_{\mathcal{N}}(\check{\mathfrak{g}}^*/\check{G}) \arrow[d] \\
AUT^{\circledast}(Bun_G(\Sigma)) \arrow[r, phantom, "\simeq"] & QC^!(Loc^{\circledast}_{\check{G}}(\Sigma)) \circlearrowleft QC^{\text{II}}(\check{\mathfrak{g}}^*/\check{G}) = \bar{\mathcal{H}}_G .
\end{tikzcd}
$$

$$(\mathcal{P}_X^{norm})^{Spec} \longleftrightarrow (\mathcal{L}_{\check{X}}^{norm})^{d}$$

Rmk Known results are mostly for $AUT_s \longleftrightarrow QC^!_{\mathcal{N}}$

But $\mathcal{L}$-sheaf $\notin QC^!_{\mathcal{N}}$.

Statement for global relative Langlands

Assume $(G, M = T^*(X, \Psi))/\mathbb{F} \longleftrightarrow (\check{G}, \check{M} = T^*(\check{X}, \check{\Psi}))/k$

split form.

Assume $\exists$ an eigenmeasure

- $\eta : G \to \mathbb{G}_m$ action of G on measure

$$Bun_G(\Sigma) \xrightarrow{\eta} Bun_{\mathbb{G}_m}(\Sigma) \xrightarrow{deg} \Sigma$$

- $\beta_X := (g-1)(\dim G - \dim X + \gamma_X)$

action of $\mathbb{G}_{gr}$ on measure.

Period side

$$\mathbb{G}_a \xleftarrow{p} Bun_G^X(\Sigma) \xrightarrow{q} Bun_G(\Sigma)$$

$$\mathcal{P}_X^{norm} := q_!(p^* \underline{AS}) \langle deg + \beta_X \rangle .$$

$\mathcal{L}$-sheaf side

$$\mathbb{A}^1[-1] \xleftarrow{\check{p}} Loc^{\check{X}}_{\check{G}} \xrightarrow{\check{q}} Loc_{\check{G}} \xrightarrow{\check{\eta}} Loc_{\mathbb{G}_m} .$$

$$\mathcal{L}_{\check{X}}^{norm} := (\check{q}_*^{\text{II}} \check{p}^{!,\text{II}}(\exp))^{\text{II}} \otimes \underbrace{\check{\mathcal{E}}_{\frac{1}{2}}}_{= \check{\eta}^*[K^{1/2}]} \langle -\beta_{\check{X}} \rangle$$

Chevalley involution ($(\mathcal{L}_{\check{X}}^{\mathrm{norm}})^d$)

$\exists!$ involution $c : G \to G$ preserving pinning $(G, B, T, (X_\alpha)_{\alpha \in \Phi^+})$
$+$ acting on torus by $t \mapsto w_0(t^{-1})$.

duality involution:

$d : G \to G = \mathrm{Ad}_{\rho(-1)} \circ c$ negate all pinning $(X_\alpha : \mathbb{G}_a \to \mathfrak{g} \ \mapsto -X_\alpha)$
$+$ acting on torus by $t \mapsto w_0(t^{-1})$.

e.g. $G = SL_n$, $c(A) = \underbrace{\begin{pmatrix} & & & 1 \\ & & -1 & \\ & \cdots & & \\ -1 & & & \end{pmatrix}}_{J} {}^tA^{-1} \cdot J^{-1}$

$d(A) = \begin{pmatrix} & & 1 \\ & \cdots & \\ 1 & & \end{pmatrix} {}^tA^{-1} \cdot \begin{pmatrix} & & 1 \\ & \cdots & \\ 1 & & \end{pmatrix}$.

Example 1 Whittaker periods:

$M = T^*_\Psi(U \backslash G) \longleftrightarrow \check{M} = pt$, $\mathrm{Loc}_{\check{G}}^{\check{X}} = \mathrm{Loc}_{\check{G}}$

$\mathcal{P}_X^{\mathrm{norm}} = \mathrm{Whit}\langle \deg + \beta_X \rangle$ $\quad \mathcal{L}_{\check{X}}^{\mathrm{norm}} = \omega\langle -\beta_{\check{X}} \rangle$

$\mathrm{Whit} \xleftrightarrow{\text{as conj'ed by geom Langlands}} \omega\langle a \rangle$

$a = (g-1)(\langle 2\rho, 2\check{\rho} \rangle - \dim U - \dim G)$.

Example 2 Spectral Whittaker model

$M = pt \longleftrightarrow \check{M} = T^*_\Psi(\check{G}/\check{U})$

const sheaf on $\mathrm{Bun}_G(\Sigma) \longleftrightarrow$ spectral Whittake sheaf on $\mathrm{Loc}_{\check{G}}(\Sigma)$

Example 3 Group case ($b_G = \dim \mathrm{Bun}_G = (g-1) \cdot \dim G$).

- $M = T^*G \circlearrowleft G \times G$,

$$\begin{array}{ccc} \mathrm{Bun}_{G\times G}^X & \longrightarrow & \mathrm{Bun}_{G \times G} \\ \| & & \| \\ \mathrm{Bun}_G & \xrightarrow{\Delta} & \mathrm{Bun}_G \times \mathrm{Bun}_G \end{array}$$

$\mathcal{P}_x = \Delta_! k \simeq \Delta_! \omega\langle -2b_G \rangle$,

$\mathcal{P}_x^{norm} = \Delta_! k \langle b_G \rangle = \Delta_! \omega \langle -b_G \rangle$.

- $\check{M} = T^*\check{G} \circlearrowleft \check{G} \times \check{G}$ by $id \times d$.

$\mathrm{Loc}_{\check{G}} \xrightarrow{\Delta} \mathrm{Loc}_{\check{G}} \times \mathrm{Loc}_{\check{G}}$

$\mathcal{L}_x = \check{\Delta}_* \omega^d$,

$\mathcal{L}_x^{norm} = \check{\Delta}_* \omega^d \langle -b_G \rangle$.

Prediction under geom Langlands:

$$\Delta_! \omega \longleftrightarrow \check{\Delta}_* \omega^d.$$

Gaitsgory's miraculous self-duality sheaf.

Serre duality sheaf

induces Serre duality on $QC^!(\mathrm{Loc}_{\check{G}})$.

Example 4 (Langlands functoriality, some cases)

Expected:

$$\begin{tikzcd} AUT(\mathrm{Bun}_{G\times H}) \arrow[r, leftrightarrow] \arrow[d] & QC^!(\mathrm{Loc}_{\check{H}\times\check{G}}) \arrow[d] \\ \mathrm{Hom}(AUT(\mathrm{Bun}_H), AUT(\mathrm{Bun}_G)) \arrow[r, leftrightarrow] & \mathrm{Hom}(QC^!(\mathrm{Loc}_{\check{H}}), QC^!(\mathrm{Loc}_{\check{G}})). \end{tikzcd}$$

Suppose $G\times H \circlearrowleft M = T^*(X, \Psi)$ $\quad \check{G}\times\check{H} \circlearrowleft \check{M} = T^*(\check{X}, \check{\Psi})$

$\mathcal{P}_x^{norm} \in AUT(\mathrm{Bun}_{G\times H}) \longleftrightarrow \mathcal{L}_{\check{x}}^{norm,d} \in QC^!(\mathrm{Loc}_{\check{G}\times\check{H}})$.

Conj

$$\begin{tikzcd} AUT(\mathrm{Bun}_H) \arrow[r, leftrightarrow, "\sim"] \arrow[d, "\mathcal{P}_x^{spec}"'] \arrow[dr, phantom, "\circlearrowright"] & QC^!(\mathrm{Loc}_{\check{H}}) \arrow[d, "\mathcal{L}_{\check{x}}^d"] \\ AUT(\mathrm{Bun}_G) \arrow[r, leftrightarrow] & QC^!(\mathrm{Loc}_{\check{G}}) \end{tikzcd}$$

__Example__ (Eisenstein series)

- $X = U \backslash G = B \backslash (G \times T)$
  $\circlearrowleft$
  $G \times T$

  $\mathrm{Bun}^{X}_{G \times T} = \mathrm{Bun}_B \longrightarrow \mathrm{Bun}_G \times \mathrm{Bun}_T$

  $\rightsquigarrow \mathrm{Bun}_T \xleftarrow{p} \mathrm{Bun}_B \xrightarrow{q} \mathrm{Bun}_G.$

- $\check{X} = \check{U} \backslash \check{G} \circlearrowleft \check{G} \times \check{T}$

  $\mathrm{Loc}_{\check{T}} \xleftarrow{\check{p}} \mathrm{Loc}_{\check{B}} \xrightarrow{\check{q}} \mathrm{Loc}_{\check{G}}$

Define $\mathrm{Eis}_* = q_* p^!$, $\mathrm{Eis}_! = q_! p^*$.

$\mathrm{Eis}_{spec} = \check{q}_* \check{p}^!$ defined by $\mathcal{L}^d$.

$$\begin{tikzcd}
\mathrm{AUT}(\mathrm{Bun}_T) \arrow[r, leftrightarrow, "\sim"] \arrow[d, "\mathrm{Eis}_!"'] & \mathrm{QC}^!(\mathrm{Loc}_{\check{T}}) \arrow[d, "\mathrm{Eis}_{spec}"] \\
\mathrm{AUT}(\mathrm{Bun}_G) \arrow[r, leftrightarrow, "\sim"] & \mathrm{QC}^!(\mathrm{Loc}_{\check{G}})
\end{tikzcd}$$

__Global GGP period (GGGPP)__

$(SL_2 \times SO_{2n}, \mathrm{std} \otimes \mathrm{std}) \longleftrightarrow (SO_3 \times SO_{2n}, \mathrm{Bessel})$

$$\begin{tikzcd}
\mathrm{SHV}(\mathrm{Bun}_{SL_2}) \arrow[d] & \text{Whittaker periods on } SL_2 \arrow[d, mapsto] \\
\mathrm{SHV}(\mathrm{Bun}_{SO_{2n}}) & \text{periods for } X = SO_{2n}/SO_{2n+1}.
\end{tikzcd}$$