Numerical Conjecture (I)

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[BZSV, §14]

<u>Setup</u> $F = \mathbb{F}_q(\Sigma)$, $k \simeq \bar{\mathbb{Q}}_\ell$ fixed, $\Gamma = \mathrm{Gal}(\bar{F}/F)$.

$\Rightarrow \sqrt{q} \in k \simeq \mathbb{C}$ chosen, $\varpi = (\varpi^{1/2})^2 : \Gamma \to k^\times$ cyclotomic.

Split forms of hyperspherical (Def 5.3.8)

$G \times \mathbb{G}_{gr}$, $M = T^*(X, \psi) / \mathbb{F}_q$,

$\check{G} \times \mathbb{G}_{gr}$, $\check{M} = T^*(\check{X}, \check{\psi}) / k$.

Assume X admits a G-eigenmeas.

L-parameter $\phi_L : \Gamma \to \check{G}$

$\rightsquigarrow$ extension $\phi_E : \Gamma \to \check{G} \times \mathbb{G}_{gr}$ by $(\phi_L, \varpi^{1/2})$.

$\check{M}^\phi$, $\check{X}^\phi$: classical fixed loci of ϕ_E.

$\forall x \in \check{M}^\phi$, $T :=$ tangent space at x

$\rightsquigarrow \quad \phi_{x,E} : \Gamma \to GL(T)$.

$\rightsquigarrow \quad L(s, T^\square)$ & $L^{\mathrm{norm}}(s, T^\square)$.

<u>Obs</u> (i) $x \in \check{X}^\phi$ isolated $\Rightarrow$ $\mathbb{G}_{gr}$-fixed

$\Rightarrow T = \bigoplus_i T^{(i)}$ by $\mathbb{G}_{gr}$-graded pieces.

$\Gamma \circlearrowright$ by $\phi^{(i)}_{x,L}$.

So $L(s, T^\square) = \prod_i L(\phi^{(i)}_{x,L}, s + \frac{i}{2})$

b/c $\phi_{x,E} = \bigoplus_i \phi^{(i)}_{x,L} \otimes \varpi^{i/2}$.

(2) $x \in \check{M}^{\phi}$ isolated.

Since $G_{gr} \curvearrowright G$ ω symplectic form by square.

$\Rightarrow$ ω pairs wts i & $2-i$.

$\Rightarrow$ $L^{norm}(s, T^{\natural}) = L^{norm}(-s, T^{\natural})$

by the FE of L^{norm} under $\begin{cases} s \longleftrightarrow 1-s \\ \text{contragredient.} \end{cases}$

e.g. $\check{G} = GL_n$, (std, W), G_{gr} acts by $\lambda \mapsto \lambda^n$,

& Γ acts by ϕ_L.

$\check{M} = W \oplus W^*$.

Then $L(s, W^{\natural}) = L(s + \frac{n}{2}, \phi_L)$,

$L(s, (W \oplus \underbrace{W^{*}\langle 2\rangle}_{\text{shift by } \lambda^2.})^{\natural}) = L(s+\frac{n}{2}, \phi_L) \cdot L(s+1-\frac{n}{2}, \phi_L^{\vee})$.

§ <u>Conjectures (tempered case)</u>

$b_G := \dim Bun_G = (g-1) \dim G$.

f everywhere unram autom form on $Bun_G(\mathbb{F}_q)$.

$$P_X(f) = \sum_{x \in Bun_G(\mathbb{F}_q)}^{\text{stacky}} P_X(x) f(x) = \int_{Bun_G} P_X \cdot f.$$

<u>Recall</u> $P_X(x) =$ "Θ-series" from the basic fcn Φ on $X(\mathbb{A})$.

<u>Eg.</u> (p.197) $X = H \backslash G$, $P_X(\mathcal{E} = \text{a } G\text{-torsor})$
$\quad\quad = \#(\text{reductions of } \mathcal{E} \text{ to } H)$,

"up to $K^{1/2}$-twist"

Convergence of $P_X(f)$: ok for G ss & f cuspidal.

otherwise: presume some regularization.

Conj π: unr tempered autom rep of $G(\mathbb{A})$ w/ parameter ϕ_L.

$\Downarrow$ unitary central char

Then can choose $f_\phi = f \in \pi^{G(\hat{\mathbb{O}})} \leftarrow$ sph

s.t. $f^d = \bar{f}$ where $\bar{f} = \mathbb{C}$-conjugacy of f

& $d :=$ duality involution

(it negates the Whittaker datum).

and s.t.

(i) f cuspidal, $\check{M} = T^*\check{X}$, $\check{X}^\phi = \{x_1, \dots, x_r\}$ finite reduced

$$\Rightarrow P_X^{norm}(f) = q^{-b_G/2} \sum_{i=1}^r L^{norm}(0, (T_{x_i}\check{X})^{\square})$$

(i)' Star-periods:

$$P_X^{*,norm}(f) = (-1)^{\dim Z_\phi} q^{-b_G/2} \cdot \sum_i L^{norm}(1, \phi^d, (T_{x_i}\check{X})^{\square})$$

fixed pts by ϕ_E^d.

where $Z_\phi := Z_{\check{G}}(\mathrm{im}\, \phi_L)$.

(ii) f cuspidal, $\check{M}^\phi = \{m_1, \dots, m_r\}$ finite reduced

$$\Rightarrow P_X^{norm}(f) = q^{-b_G/2} \sum_{i=1}^r \sqrt{L(0, (T_{m_i}\check{M})^{\square})}$$

Here $\forall L^{norm}(-) \in \mathbb{R}_{>0}$, $\exists$ choice of $\sqrt{\ }$ making equality holds.

invariant under $Z_\phi \curvearrowright$ these m_i's.

<u>Examples</u> (1) Conj $\Rightarrow$ normalized Whittaker P of $f = q^{-b_G/2}$.

by taking $X = U\backslash G$, $\check{X} = pt = \check{M}$.

(2) Conj $\Rightarrow \int_{\mathrm{Bun}_G(\mathbb{F}_q)} |f|^2 = |Z_\phi| \cdot L(1, \mathrm{Ad}, \check{\mathfrak{g}})$

whenever f cuspidal + G semisimple.

by taking $X = G \circlearrowleft G \times G \Rightarrow$ LHS.

$\check{X} = \check{G} \times \check{G} \circlearrowleft \check{G}$

(ϕ, ϕ)

$\Rightarrow$ fixed pts $= Z_\phi = Z_G(\operatorname{im}\phi)$

+ details $\rightsquigarrow$ get the RHS of conj.

§ Nontempered case

Arthur parameter: $\phi_A : \Gamma \times SL_2 \to \check{G}(k)$

s.t. $\phi_A|_\Gamma$ is pure of wt 0

(i.e. Frob eigenvals are all Weil integers of wt 0).

$\phi_A \rightsquigarrow \phi_L = \phi_A \circ (id_\Gamma, \begin{pmatrix} \varpi^{1/2} & \\ & \varpi^{-1/2} \end{pmatrix}) : \Gamma \to \check{G}$.

$\rightsquigarrow \phi_E : \Gamma \to \check{G} \times \mathbb{G}_{gr}$.

Lem 14.3.2 ϕ_A as above, then $\overline{\operatorname{im}\phi_E}^{Zar} \supset \operatorname{im}(a, id)$

where $(a, id) : \mathbb{G}_m \to \check{G} \times \mathbb{G}_{gr}$

$a : \mathbb{G}_m \longrightarrow \check{G}$

$\lambda \longmapsto \phi_A(1, \begin{pmatrix} \lambda & \\ & \lambda^{-1} \end{pmatrix})$.

Slogan Understand periods in terms of Slodowy slices of $\check{M}$:

$$\begin{tikzcd} \check{M}_{slice} \arrow[r, hook] \arrow[d] \arrow[dr, phantom, "\square"] & \check{M} \arrow[d, "\mu"] \\ f + Z_{\check{\mathfrak{g}}}(e) \arrow[r, hook] & \check{\mathfrak{g}}^* \end{tikzcd}$$

(h, e, f): $\mathfrak{sl}_2$-triple attached to $\phi_A|_{SL_2}$ fixing $\check{\mathfrak{g}} \simeq \check{\mathfrak{g}}^*$.

Rmk $(a, id) \circlearrowright (Ad, \square)$ preserves $f + Z_{\check{\mathfrak{g}}}(e)$

since $a(\lambda) f = \lambda^{-2} \cdot f$ (See §3.1 on graded Hamiltonian spaces.)

& $\underset{\mathbb{G}_{gr}}{\lambda \cdot} f = \lambda^2 \cdot f$

$\Rightarrow \check{M}_{slice}$ is Γ-inv.

* <u>Theory of Slodowy slices</u> (extends Kostant sections)

$\check{G} \times (f + Z_{\check{\mathfrak{g}}}(e)) \xrightarrow{act} \check{\mathfrak{g}}^*$ is smooth.

[BZSV, after Lem 3.4.9]

Moreover, $\check{M}_{slice} \simeq \check{M} /\!/\!/_f\, U$, U = unipotent def'd by (h, e, f).

$\Rightarrow \check{M}_{slice}$ is sm by general theory of Hamil red'n.

Also, unproven lem $\Rightarrow \check{M}^{\Gamma}_{slice} \to \{f\} \subset \check{\mathfrak{g}}^*$.

Conj Take $(M, \check{M})$, M polarized $= T^*X$.

assume ϕ_A discrete, f_ϕ parametrized by ϕ_A (unr).

$f_\phi^{d} = \overline{f_\phi}$.

$\check{M}^{\phi}_{slice} = \{m_1, \dots, m_r\}$ reduced.

<u>Then</u> $P_X^{norm}(f_\phi) = q^{-b_0/2} \cdot \sum_i \sqrt{L^{norm}(0, T_{m_i} \check{M}^{\Omega}_{slice})}$

+ same interpretations as before.

<u>Rmk</u> Assume G ss. Then

$$\int_{Bun_G(\mathbb{F}_q)} |f_\phi|^2 \overset{conj}{=} |Z_\phi| \cdot L(1, \mathrm{Ad}, Z_{\check{\mathfrak{g}}}(e)).$$

If $f_\phi = \mathbb{1}$ ((h, e, f) = principal),

we get a formula for Tamagawa numbers.

Rmk In "some" cases (possibly $X = H\backslash G$),

$\check{M}_{\text{slice}} \simeq$ universal centralizer scheme of sth.

§ From geometric to numerical (§14.7)

<u>Heuristics</u> $M = T^*(X, \Psi)$, $\check{M} = T^*\check{X}$.

$$\begin{array}{ccc} \mathcal{P}_X^{\text{spec}} := \text{the spectral proj of } \mathcal{P}_X & & \\ \Big\updownarrow \text{geom conj} & & \text{IS conj} \\ & & \text{right adjoint of } \mathrm{Shv}_{\mathrm{Nilp}} \subset \mathrm{Shv}. \\ \mathcal{L}_{\check{X}} : \text{ L-sheaf}. & & \end{array}$$

Let $\phi \in \mathrm{Loc}_{\check{G}}(k)$.

Let $\mathcal{F}_\phi$ be pure, self-dual perverse Hecke eigensheaf. cuspidal.

Assume $\phi|_{\pi_1^{\text{geom}}(\Sigma)}$ has a single fixed pt on $\check{X}$ (reduced).

Same ϕ^d as before.

(<u>Rmk</u> Example w/ multiple x_i, $L(-)$.
See X. Wan's (万小磊) PhD thesis.)

Then $\mathcal{F}_\phi \rightsquigarrow f : \mathrm{Bun}_G(\mathbb{F}_q) \to k$

$\quad \ \ \text{IS} \qquad\qquad \Vert$

$\mathbb{D}\mathcal{F}_\phi \rightsquigarrow \tilde{f} : \mathrm{Bun}_G(\mathbb{F}_q) \to k$, in fact $\tilde{f} = \bar{f}$ (using $k \simeq \mathbb{C}$).

$$\mathrm{Hom}(\mathcal{F}_\phi, \mathcal{P}_X^{\text{norm}}) \underset{\uparrow}{\simeq} \mathrm{Hom}(\mathcal{F}_\phi, \mathcal{P}_X^{\text{norm, Spec}})$$

conj on projector $\qquad$ IS geom conj

$$\mathrm{Hom}(\delta_\phi, \mathcal{L}_{\check{X}}^{\text{norm}, d})$$

skyscraper.

Take $\mathrm{Tr}(\mathrm{Frob})$:

LHS: use Lem 2.4.1 to get $\displaystyle\int_{\mathrm{Bun}_G(\mathbb{F}_q)} f_\phi \cdot \overbrace{\mathcal{P}_X^{*, \text{norm}}}^{\exists \text{ a } \mathbb{D} \text{ in } 2.4.1.}$

RHS: apply (11.33) to get $q^{-bc/2} \cdot L^{norm}(1, \phi^d, T^{\vee\square})$

$T :=$ tangent at the unique fixed pt.

& equality is up to $(-1)^{\dim Z_\phi}$.

For usual or !-periods:

Want $\mathrm{Hom}(\mathcal{P}_X^{norm}, \mathcal{F}_\phi)$.

<u>Assumption</u> $\mathrm{Hom}(\mathcal{P}_X^{norm, Spec}, \mathcal{F}_\phi) \simeq \mathrm{Hom}(\mathcal{L}_X^{norm}, \delta_\phi)$.

By taking Tr of Frob,

$$\int P_X^{norm} \tilde{f} = q^{\underbrace{-bc/2} \cdot} L^{norm}(0, \phi^d, T^\square)$$

see (11.35).

Now $\tilde{f}$ should be parametrized by ϕ^d.

(Look at [V. Lafforgue] or [Li, Contragredients].

Replace $\tilde{f}$ by f $\Rightarrow$ $\int P_X^{norm} f = q^{-bc/2} \cdot L^{norm}\underbrace{(0, T^\square)}_{\text{w.r.t. } \phi_E.}$

ϕ^d by ϕ

In general, left adjoint of $Shv_{Nilp} \hookrightarrow Shv$ need not exists.

but may expect some "interpretation". see § 12.4.2.