Introduction

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§ Motivation

(1) Classical motivation:

Hecke: π rep of $PGL_2(\mathbb{A}_{\mathbb{Q}})$

$\rightsquigarrow L(\frac{1}{2}+s, \pi) \approx \int_{\begin{pmatrix} * & \\ & 1\end{pmatrix}} f(h)\cdot|h|^s dh$

period integral

$\overset{\text{unfold}}{\rightsquigarrow}$ Whittaker functionals $\forall v$ place,

i.e. $W: PGL_2(\mathbb{Q}_v) \longrightarrow \mathbb{C}$ with some $\psi: \mathbb{Q}_v \to \mathbb{C}$

s.t. $W(\begin{pmatrix} 1 & t \\ & 1\end{pmatrix} g) = \psi(t) W(g)$, $\forall t \in \mathbb{Q}_v$.

Generalization: GGP conjs, etc.

(2) Local motivation:

F local field, G conn red grp $/F$,

X "reasonable" homogeneous G-space

s.t. $X(F) \circlearrowleft G(F)$ assuming $\exists$ invariant measure.

Spectral decomp (in abstracts)

$$L^2(X) = \int^{\oplus}_{\mathrm{Unit}(G)} \mathcal{H}_\pi \, d\mu_{\mathrm{Plancherel}}(\pi).$$

$\mathcal{H}_\pi$: π-isotypic unitary rep of $G(F)$

<u>Goal</u> Make this explicit.

E.g. What is $\mu_{\mathrm{Plancherel}}$?

Reformulation/Refinement: $\Phi_i \in \mathcal{S}(X)$ Schwartz func, $i=1,2$

$\rightsquigarrow L^2$ inner product $(\Phi_1 | \Phi_2)_{L^2(X)} = \int J_\pi(\Phi_1, \Phi_2)\, d\mu_{\mathrm{Plancherel}}(\pi)$

where $J_\pi : \mathcal{S}(X) \otimes \overline{\mathcal{S}(X)} \xrightarrow{\text{quotient}} (\mathcal{S}(X))_{\pi\text{-iso}} \otimes (\overline{\mathcal{S}(X)})^\vee_{\pi\text{-iso}}$

$\xrightarrow{G\text{-invariant}} \mathbb{C}$

E.g. $G = H \times H$, $X = H \circlearrowleft H \times H$.

$$L^2(H) \overset{H\text{-}C}{=} \int^{\oplus}_{\mathrm{Temp}(H)} \overbrace{\sigma \boxtimes \sigma^\vee}^{\pi} \, d\mu_{\mathrm{Plancherel}}(\pi)$$

- Everything is explicit here.

Now Assume X is sperical

i.e. $\exists$ open B-orbit in X (B Borel subgrp of G).

say G is split here.

In [SV17], it is conjectured:

(i) Can construct $\check{G}_X$ together with $\check{G}_X \times Sl_2(\mathbb{C}) \longrightarrow \underset{\substack{\uparrow \\ \text{dual grp of } G}}{\check{G}}$

(ii) $(\Phi_1 | \Phi_2)_{L^2(X)} = \int_{W'_F \xrightarrow{\varphi} \check{G}_X} J_\varphi^{\mathrm{Plancherel}}(\Phi_1, \Phi_2) \cdot d\nu_X(\varphi)$

with · ν_X = the "standard measure" on such φ.

· $J_\varphi^{\mathrm{Plancherel}} : \mathcal{S}(X) \otimes \overline{\mathcal{S}(X)} \longrightarrow \Pi_\varphi \otimes \overline{\Pi_\varphi} \longrightarrow \mathbb{C}$

· $\Pi_\varphi = \bigoplus$ (irreps of Arthur-packet) (with multiplicities ?)

attached to $W'_F \times SL_2 \longrightarrow \check{G}_X \times SL_2 \longrightarrow \check{G}$.

(3) Global motivation

F global field, $X \circlearrowleft G$ spherical

Roughly, $\mathrm{RTF}_X(\Phi_1, \Phi_2) = \int_{\underset{\substack{\uparrow \\ L\text{-parameters into } \check{G}_X.}}{\varphi}} J_\varphi^{\mathrm{global}}(\Phi_1, \Phi_2)\, d\nu_X(\varphi)$

Here $\Phi_1, \Phi_2 \in \mathcal{S}(X(\mathbb{A}))$, $\mathbb{A} = \mathbb{A}_F$.

$$J_\varphi^{\text{global}} \text{ expected} \approx |\text{Periods of autom forms}|^2 \times (\text{L-value})^{-1}$$
$$J_\varphi^{\text{global}} \simeq \prod_v J_{\varphi_v}^{\text{Plancherel}}$$

§ <u>Relative trace formula</u> (modulo convergence)

Let the theta series

$$\Theta_\Phi^X(g) = \sum_{x \in X(F)} \Phi(xg), \quad g \in G(\mathbb{A})$$

$$\Theta^X : \mathcal{S}(X(\mathbb{A})) \longrightarrow \mathrm{Fcn}(G(F)\backslash G(\mathbb{A}), \mathbb{C}) \quad G(\mathbb{A})\text{-equivariant.}$$

X_1, X_2 sperical varieties

Then $$RTF(\Phi_1, \Phi_2) = \int_{G(F)\backslash G(\mathbb{A})} \Theta_{\Phi_1}^{X_1} \overline{\Theta_{\Phi_2}^{X_2}}, \quad \Phi_i \in \mathcal{S}(X_i(\mathbb{A}))$$

(assume G-semisimple).

Suppose $X_i = H_i\backslash G$. More common: $\forall f \in \mathcal{S}(G(\mathbb{A}))$,

$$RTF(f) = \int_{H_1(F)\backslash H_1(\mathbb{A})} \int_{H_2(F)\backslash H_2(\mathbb{A})} \sum_{\gamma \in G(F)} f(x\gamma y^{-1})\, dx\, dy.$$

$\Rightarrow$ Same result if $X_i(\mathbb{A}) = H_i(\mathbb{A})\backslash G(\mathbb{A})$ & $X_i(F) = H_i(F)\backslash G(F)$.

<u>Idea</u> RTF should be a linear functional on $\mathcal{S}$ of

(forget about complex conj) $$\big(H_1(\mathbb{A})\backslash G(\mathbb{A}) \times H_2(\mathbb{A})\backslash G(\mathbb{A})\big)/G(\mathbb{A}).$$
$$\text{is } H_1(\mathbb{A})\backslash G(\mathbb{A})/H_2(\mathbb{A})$$

Have

$$\begin{tikzcd}
\mathcal{S}\big(\big[\big(H_1(\mathbb{A})\backslash G(\mathbb{A}) \times H_2(\mathbb{A})\backslash G(\mathbb{A})\big)/G(\mathbb{A})\big]\big) & \\
\mathcal{S}\big(H_1(\mathbb{A})\backslash G(\mathbb{A}) \times H_2(\mathbb{A})\backslash G(\mathbb{A})\big) \arrow[u, two heads] & \mathcal{S}\big(\big[H_1(\mathbb{A})\backslash G(\mathbb{A})/H_2(\mathbb{A})\big]\big) \arrow[ul, "\simeq"'] & \Phi_1 \otimes \Phi_2 \\
\mathcal{S}(G(\mathbb{A}) \times G(\mathbb{A})) \arrow[u] \arrow[r, two heads] & \mathcal{S}(G(\mathbb{A})) \arrow[u, two heads] & \\
f_1 \otimes f_2 \arrow[r, mapsto] & f = f_1 * f_2^\vee \arrow[ruu, mapsto, bend right] &
\end{tikzcd}$$

§ Unramified local situation

$\rightsquigarrow \Phi_X \in \mathcal{S}(X)$ "basic" function
$\quad\quad \overset{\shortparallel}{\mathbb{1}_{X(\mathcal{O})}}$ if X is smooth.

In the geometric / unramified local setting,
we impose structures

replacing $L^2(X)$ by $C_c^\infty(X_F/G_{\mathcal{O}})$ where $\mathcal{O} \subset F$.
$C_c^\infty(G(F)\backslash G(\mathbb{A}))$ by $C_c^\infty(G(F)\backslash G(\mathbb{A})/G(\hat{\mathcal{O}}))$
with Hecke alg action.

<u>So</u> (unramified local) $(\Phi_X \mid h \cdot \Phi_X)_{L^2(X)}$
h: Hecke operator

(global geom) $RTF(\prod_v \Phi_{X_v} \mid h \cdot \prod_v \Phi_{X_v})$
h: global Hecke operator

<u>Rmk</u> We also have to accomodate the case $C_c^\infty(H(F)\backslash G(F), \psi)$
(say F local)
where $\psi: H(F) \longrightarrow \mathbb{C}^\times$ subject to the conditions
(e.g. "genericity")

- This is the Whittaker induction setting.

§ Tiers of Langlands

<u>"Arithmetic part"</u>

$\#$ TF of G / global F

v.s. $\{\text{Autom forms}\} = \mathcal{A}(G)$ / global F

cat $\mathrm{Rep}\, G(F)$ / local F

"Geometric part"

Shv Geometrizations of TF.

Cat Cat of custom sheaves / Bun_G (F global)

2-Cat Categorical rep of $LG = G_F / G_O$ (F local)

Rmk L-functions doesn't appear in this picture.

§ Relative Langlands

"Arithmetic part"

\# RTF^{X_1, X_2} / global F

versus Θ^X_Φ for various Φ $\circlearrowleft$ Hecke operator

\# $(\cdot|\cdot)_{L^2(X)}$ / local F

versus $\Phi \in \mathcal{S}(X)$ for various Φ $\circlearrowleft$ Hecke operator.

"Geometric part"

Replace Θ^X_Φ by X-period sheaves / $\mathrm{Shv}(\mathrm{Bun}_G)$

Φ by categorical rep of LG / $\mathrm{Shv}(LX) \circlearrowleft LG$

Also RTF algebra, Plancherel algebra.

Endomorphisms of "basic X-period shvs"

or "basic X-period shvs on LX/LG"

with Hecke operators inserted.

$\Big\}$ need ∞-Cat.

See p.9 of [BZSV].

§ Spectral side

(favorable) X spherical G-var $\rightsquigarrow$ T^*X Hamiltonian G-space

$\cap$

(good) M Hamiltonian G-space

hyperspherical

$\updownarrow$

$M^\vee$ Hamiltonian $G^\vee$-space

(not necessarily a $T^*(-)$).

Expectation	Local	Global
geometric	Coh sheaves on $M^\vee/G^\vee$	"L-sheaves" on the spectral side
arithmetic	In §9.4 spherical fcns on X	L-fcn appeared in $RTF^{X,X}$

Innovation in [BZSV].

Meta-Conjecture The relative setting

automorphic $\longleftrightarrow$ Spectral

§ Final remarks

$\exists$ Plenty examples of $X, M, M^\vee$, L-fcns

Explainations from TQFT

$\rightsquigarrow$ motivates the work of $M, M^\vee$.