Dual of spherical varieties

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Some history

(1920s) Weyl: Weyl grp for symmetric vars.

(1980s, 1990s) Brion, Knop:

Weyl grp for all spherical vars
+ some root systems.

(2006) Gaitsgory & Nadler:

dual for affine spherical vars

(2010s) SV: dual for spherical vars (+ mild conditions)
using root datum with $G_X^\vee \times SL_2 \overset{\iota}{\hookrightarrow} G^\vee$.

(2017) KS: $G_X^\vee$ for general X + existence of ι.

Notation $k = \bar{k}$, char $k = 0$.

G/k red grp, $G \supset B \supset A$ (B Borel, A max torus) $\rightsquigarrow$ Φ roots $\supset \Phi^+$ pos roots $\supset \Delta$ simple roots.

U = unip rad $= R_u(G)$

Polarization of hyperspherical varieties

Let M hyperspherical hamiltonian G-space.

- M has a distinguished polarization

$\Rightarrow M = T^*(X, \Psi)$ for X spherical

s.t. the stabilizer of a general pt in B is conn.

Spherical vars

Def $X/k \curvearrowleft G$. Say X spherical if

X normal + X has an open B-orbit.

E.g. (1) If $G = A$ then spherical varieties $\Leftrightarrow$ normal toric varieties

(2) Symmetric vars $\theta : G \to G$ involution

then $G^\theta \backslash G$ is spherical.

- Group case: $G = H \times H$, $\theta(a,b) = (b,a)$

$$\Rightarrow G^\theta = \Delta H, \quad \begin{array}{ccc} G^\theta \backslash G & \longrightarrow & H \\ (h_1, h_2) & \longmapsto & h_1^{-1} h_2 \end{array}$$

(3) Horospherical vars: $X = H \backslash G$, s.t. $H \supseteq U$

(4) Flag varieties.

Thm If X is spherical, then X has only fin many B-orbits.

Idea Step1 Reduce to X homogeneous.

Lem X has fin many G-orbits & each of them is spherical.

- Y = union of G-orbits

$X_0 = X - \underbrace{\bigcup D}$

D prime divisor, B-stable s.t. $D \not\supseteq Y$.

$\Rightarrow X_0$ affine B-stable

and $X_0 \cap Y$ is the unique closed orbit.

Step2 Reduce to horospherical case.

Produce $\mathfrak{X} \to Z$ family of spherical vars

with gen fiber $\simeq X$ & special fiber horo.

(by semi-continuity.)

Step 3 Bruhat decomp for $U \backslash G$.

Universal Cartan of X

X° = the open B-orbit of X.

$P(X)$ = max standard parabolic s.t. $X^\circ P(X) = X^\circ$.

Write $P(X) = L(X) \ltimes U(X)$.

$U(X)$ acts on X° freely $\rightsquigarrow$ $X^\circ / U(X) \curvearrowleft L(X)$.

this action factors through $L(X) \twoheadrightarrow A_X$.

Def Universal Cartan of X is A_X.

Let $k(X)^{(B)}$:= multiplicative grp of B-eigen rational funcs over X.

i.e. $\exists\ \chi_f \in \mathrm{Hom}(B, \mathbb{G}_m)$ s.t. $f(xb) = \chi_f(b) \cdot f(x)$.

Suppose $\chi_{f_1} = \chi_{f_2} \Rightarrow f_1/f_2$ is const on X.

$$1 \longrightarrow k^* \longrightarrow k(X)^{(B)} \longrightarrow \mathcal{X}(X) \longrightarrow 1$$

$$\Rightarrow X^*(A_X) = \mathcal{X}(X).$$

Similarly, define $\mathcal{X}(Y)$ for any B-orbit Y

$$rk(Y) := rk(\mathcal{X}(Y)).$$

Little Weyl grp

Knop's action Let $B(X)$ = set of all B-orbits inside.

Want : $W = W_G \curvearrowright B(X)$.

Let $\alpha \in \Delta$, s_α the reflection given by α.

$Y \in B(X)$, want to define $s_\alpha Y$.

$Y P_\alpha/R(P_\alpha) \curvearrowleft P_\alpha^{ad} = PGL_2$ is PGL_2-homogeneous spherical.

$H\backslash PGL_2$ spherical $\iff$ $H \curvearrowright PGL_2/B = \mathbb{P}^1$ has finite orbits.

<u>Def</u> (Y, α) is of types:

(U) $YP_\alpha/R(P_\alpha) \simeq SU\backslash PGL_2$, $S \subseteq \mathbb{G}_m$.

only 2 orbits: $\{0\}$, $\mathbb{P}^1 - \{0\}$

s_α switches them.

(N) $YP_\alpha/R(P_\alpha) \simeq N(\mathbb{G}_m)\backslash PGL_2$

only 2 orbits: $\{0\} \cup \{\infty\}$, $\mathbb{A}^1 - \{0\}$

s_α s_α s_α fixes both orbits

(T) $YP_\alpha/R(P_\alpha) \simeq \mathbb{G}_m\backslash PGL_2$

3 orbits: $\{0\}$, $\{\infty\}$, $\mathbb{A}^1 - \{0\}$

s_α s_α fixes open orbit & switches closed orbits.

(G) $YP_\alpha/R(P_\alpha) \simeq PGL_2\backslash PGL_2$

only 1 orbit $\mathbb{P}^1$ fixed by s_α.

<u>Thm</u> The above extends to action of $W \curvearrowright \mathcal{B}(X)$.

<u>Idea</u> $G \times X \longrightarrow X$, $(g, x) \longmapsto gx$.

$$\begin{array}{cccc} \rightsquigarrow & IC(BwB) & Shv(B\backslash G/B) \curvearrowright & Shv(B\backslash X) \\ & \downarrow & \downarrow & \downarrow \\ & [w] & \mathbb{C}[W] & M \end{array}$$

$f_{Y,\rho} := IC(Y, \rho)$, Y B-orbit

ρ equiv loc sys on Y.

$W_{(x)} :=$ stabilizer of $\mathring{X}$ under Knop's action.
$\overset{\cap}{W}$

<u>note</u> $X^\circ P(x) = X^\circ \Rightarrow W_{L(x)} \subseteq W_{(x)}$ & $W_{(x)} \curvearrowright X^*(A_x)$
with $W_{L(x)}$ acts trivially.

<u>Thm</u> The quotient $W_X := W_{(x)}/W_{L(x)}$ $(\pi : W_{(x)} \twoheadrightarrow W_X)$
acts on $X^*(A_x)$ faithfully as a reflection grp
for $\alpha \in W_X$ the min length representative in $\pi^{-1}(\alpha)$
giving a splitting of π.
$\Rightarrow W_{(x)} \simeq W_X \ltimes W_{L(x)}$.

<u>Spherical roots</u>

Let V be the set of all G-inv discrete $\mathbb{Q}$-valued valuations on $k(X)$.
For any $v \in V$ and $\chi \in \mathcal{X}(X) = X^*(A_x)$,
$v(f_\chi)$ depends only on χ.
$\rightsquigarrow$ induce $\mathcal{V} \longrightarrow \mathrm{Hom}(X^*(A_x), \mathbb{Q}) = X_*(A_x)_{\mathbb{Q}}$.

<u>Thm</u> $\mathcal{V} \to \mathrm{Hom}(X^*(A_x), \mathbb{Q}) = X_*(A_x)_{\mathbb{Q}}$ is injective.
- image is a f.g. convex cone
- image contains the image of negative Weyl Chamber
$X_*(A)_{\mathbb{Q}} \longrightarrow X_*(A_x)_{\mathbb{Q}}$
- $\mathcal{V}$ is fundamental domain for W_X.

Let $\mathcal{V}^\perp := \{\alpha \in X^*(A_x)_{\mathbb{Q}}$ s.t. $\langle \alpha, v\rangle \leq 0 \ \forall v \in \mathcal{V}\}$.

Def Δ_X is the generator of extremal rays of $\mathcal{V}^{\perp}$
s.t. $\sigma \in \Delta_X$ is primitive in $\mathbb{Z}\Delta$.

Thm Any $\sigma \in \Delta_X$ is
- either a simple root of G
- or $\sigma = \sigma_1 + \sigma_2$, where σ_1, σ_2 strongly orthogonal roots $\in \Phi$.

At first,

$$X^{\circ} P_\sigma / R(P_\sigma) = \begin{cases} \mathbb{G}_m \backslash PGL_2, & \sigma \in X^*(A_X) \quad \text{(type T)} \\ N(\mathbb{G}_m) \backslash PGL_2, & \sigma \notin X^*(A_X) \ \& \ 2\sigma \in X^*(A_X) \quad \text{(type N)}. \end{cases}$$

If $T^*(X)$ hyperspherical, then Case 2 won't happen.
In Case 2, if we require $\sigma_1^\vee - \sigma_2^\vee = \delta_1^\vee - \delta_2^\vee$ $(\delta_1, \delta_2 \in \Delta)$,
then σ_1, σ_2 are unique for $\sigma \in X^*(A_X)$,
$\sigma_1^\vee, \sigma_2^\vee$ have the same image in $X_*(A_X)$.
Set $\sigma^\vee := \sigma_1^\vee|_{X^*(A_X)}$.

Thm If no spherical roots of Type N, then
$(X^*(A_X), \Delta_X, X_*(A_X), \Delta_X^\vee)$ ← can def $G_X^\vee$ using this dual
gives a based root datum with Weyl grp W_X.

Distinguished morphisms

Want $G_X^\vee \longrightarrow G^\vee$, $G_X^\vee \times SL_2 \longrightarrow G^\vee$.
Here $G^\vee$ is a pinned red grp,
i.e. fix $e_{\alpha^\vee} \in \mathfrak{g}_{\alpha^\vee}$ for each $\alpha^\vee$ ⇝ principle SL_2

$\rightsquigarrow \; \iota : SL_2 \to L(X)^\vee$.

For any $\sigma \in \Delta_X$,

$$\mathfrak{g}^\vee \supseteq \mathfrak{g}^\vee_{\sigma^\vee} := \begin{cases} \mathfrak{g}^\vee_{\sigma^\vee} & \text{if } \sigma \in \Delta \\ \mathbb{C}[e_{\sigma_1^\vee} - e_{\sigma_2^\vee}] & \text{if } \sigma_1^\vee = \delta_1^\vee,\ \sigma_2^\vee = \delta_2^\vee \\ V & \text{else} \end{cases}$$

where $V =$ the unique 1-dim subspace spanned by $\mathfrak{g}^\vee_{\sigma_1^\vee}, \mathfrak{g}^\vee_{\sigma_2^\vee}$ s.t. it commutes with ι.

Thm $\exists\, \varphi : G_X^\vee \times SL_2 \to G^\vee$ s.t.

(1) $\varphi|_{SL_2} = \iota$

(2) $\varphi|_{A_X^\vee}$ is dual to $A \twoheadrightarrow A_X$

(3) For each $\sigma \in \Delta_X$, $\varphi_\cdot(\mathfrak{g}^\vee_{X,\sigma^\vee}) = \mathfrak{g}^\vee_{\sigma^\vee}$

(4) If φ satisfies (1)–(3) then φ is unique up to $A_X^\vee$ conj.

(5) $\ker\varphi$ is finite

If T^*X is hyperspherical then φ is injective.