Towards hyperspherical duality

Liang Xiao

<u>Goal today</u> §5 Explain (?) the conjectural duality

$$(G, M) \longleftrightarrow (G^\vee, M^\vee)$$

- * anomally (related to Mp_{2n} vs Sp_{2n})
  $\shortparallel$
  breaking symmetry
- * duality statement revisit
- * extension to Spec $\mathbb{Z}$.

§1 Anomally (BZSV: def'n is provisional)

<u>Motivating lemma</u>

Let F local field of res char $\neq 2$

V symp space $/F$.

$\rightsquigarrow Mp(V)(F) \xrightarrow{2:1} Sp(V)(F)$ <u>not algebraic</u>

(So shall take F-pts)

$H \leq Sp(V)$ an F-algebraic subgrp.

Suppose that $\exists$ a char $\theta : H \to \mathbb{G}_m$ s.t.

$$\underset{\substack{\uparrow \\ \text{2nd Chern class}}}{c_2(V)} = c_1(\theta)^2 \quad \text{in } H^4_{et}(\underset{\substack{\uparrow \\ \text{classifying space of H-cples} \\ \text{viewed as a stack}}}{BH}, \mathbb{Z}/2\mathbb{Z})$$

Then

$$\begin{tikzcd}
H(F) \times \mathbb{Z}/2\mathbb{Z} \arrow[r, "\exists"] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & Mp(V)(F) \arrow[d, "2:1"] & \leftarrow \text{not alg} \\
H(F) \arrow[r, hook] \arrow[ur, dashed] & Sp(V)(F) & \leftarrow \text{alg (even if)}
\end{tikzcd}$$

Explaination $BH = [*/H]$, $\mathrm{Coh}([*/H]) = \mathrm{Rep}(H)$

If given $\theta : H \to \mathbb{G}_m$

$\rightsquigarrow [*/H] \longrightarrow [*/\mathbb{G}_m]$.

Can consider

$$\begin{tikzcd}
\theta^* : \mathrm{Coh}([*/\mathbb{G}_m]) \arrow[r] \arrow[d, equal] & \mathrm{Coh}([*/H]) \arrow[d, equal] \\
\mathrm{Rep}(\mathbb{G}_m) \arrow[r] & \mathrm{Rep}(H).
\end{tikzcd}$$

Will define a cohomological class in $H^2_{et}([*/\mathbb{G}_m], \mathbb{Z}/2\mathbb{Z})$

(called univ Chern class)

and pull it back to $H^2_{et}([*/H], \mathbb{Z}/2\mathbb{Z}) \rightsquigarrow c_1(\theta)$.

Example

$N \geq 2$ $[\mathbb{A}^{N+1} - \{0\} / \mathbb{G}_m] \simeq \mathbb{P}^N$, $\quad \underset{\text{open}}{\mathbb{A}^{N+1} - \{0\}} \hookrightarrow \mathbb{A}^{N+1} \hookleftarrow \underset{\text{closed}}{\{0\}}$.

$$\begin{tikzcd}
 & & H^2_{et}([*/\mathbb{G}_m]) \arrow[d, equal] & & \\
0 \arrow[r, phantom, "="] & H^2_{[\{0\}/G]}([\mathbb{A}^{N+1}/G]) \arrow[r] & H^2_{et}([\mathbb{A}^{N+1}/\mathbb{G}_m]) \arrow[r] & H^2_{et}(\overbrace{[\mathbb{A}^{N+1} - \{0\}/\mathbb{G}_m]}^{\mathbb{P}^N}) \arrow[r, phantom, "\ni"] \arrow[ddll] & c_1(\mathcal{O}(1)) \\
 & \text{supported at } \{0\} \arrow[u] & & & \\
 & H^3_{[\{0\}/G]}([\mathbb{A}^{N+1}/G]) \arrow[r] \arrow[d, equal] & \cdots & & \\
\text{by purity} \arrow[uuu, bend left] \arrow[r] & 0 & & &
\end{tikzcd}$$

For G red grp $/\mathbb{C}$, M = symp G-var.

$\rightsquigarrow c_2 := c_2(TM) \in H^4_G(M, \mathbb{Z})$

Definition 5.1.2 Say M is

- strongly anomally free if $c_2 \equiv 0 \bmod 2$
- anomally free if $\exists\, \beta \in H^2_G(M, \mathbb{Z})$ s.t. $c_2 \equiv \beta^2 \bmod 2$

(need $\beta \in H^2_G(M, \mathbb{Z})$, not just $H^2_G(M, \mathbb{Z}/2\mathbb{Z})$.)

Expectation Hyperspherical duality should wrote for anomally free (G,M)'s.

Structure thm (Recall)

$$\left\{\begin{matrix}\text{hyperspherical Hamiltonian}\\ G\text{-var } (G,M)\end{matrix}\right\} \overset{??}{\longleftrightarrow} \left\{\begin{matrix} H\times SL_2 \overset{\eta}{\hookrightarrow} G \\ H \xrightarrow{\lambda} Sp(S)\end{matrix}\right\}$$

$$\eta(sl_2) \overset{ad}{\circlearrowright} \mathfrak{g} = \mathfrak{j} \oplus \bar{\mathfrak{u}} \oplus \mathfrak{u}_0 \oplus \mathfrak{u}$$

$\mathfrak{j}$ ← Centralizer of entire $\eta(sl_2)$.

wt of $\bar{\mathfrak{u}}$: <0, of $\mathfrak{u}_0$: 0, of $\mathfrak{u}$: >0

$$\mathfrak{u}_+ := \bigoplus_{i\geq 2} \mathfrak{g}_i .$$

$$(\eta,\lambda) \rightsquigarrow \tilde{S}$$

$$M := h\text{-ind}_{HU}^{G}(\underbrace{S\times(\mathfrak{u}/\mathfrak{u}_+)_f}_{=\tilde{S}})$$

$\mu : S \longrightarrow \mathfrak{g}^*$ moment map:

given $s\in S$, $h\in\mathfrak{h}^*$, $\mu(v)(h) := \frac{1}{2}\omega(hv,v)$.

$$\mu : \underset{\overset{\shortparallel}{\mathfrak{u}_1}}{(\mathfrak{u}/\mathfrak{u}_+)} \xrightarrow{\kappa_f} (\mathfrak{u}/\mathfrak{u}_+)^* \xrightarrow{\xi \mapsto \xi+\eta(f)} \mathfrak{u}^*$$

$(\mathfrak{u}_1\times\mathfrak{u}_1 \to \mathbb{C}$ via $(X,Y)\mapsto \langle \eta(f), [X,Y]\rangle)$.

$$M \simeq (\tilde{S}\times_{(\mathfrak{h}\oplus\mathfrak{u})^*}(\mathfrak{g}^*\times G))/HU.$$

⇝ Lagrangian correspondence

$$\begin{tikzcd}
 & M^+ := \tilde{S}\times_{(\mathfrak{h}+\mathfrak{u})^*}\mathfrak{g}^* \arrow[dl, "\ker(\mathfrak{g}^*\to(\mathfrak{h}+\mathfrak{u})^*)\text{-affine bundle}"'] \arrow[dr] & \\
\tilde{S} \arrow[d, phantom, sloped, "\subset"] & * \arrow[dl] \arrow[dr] & M \circlearrowleft G\text{-action} \arrow[d, phantom, sloped, "\supset"] \\
\{0\} & & M_0 \leftarrow \exists!\text{ closed orbit} \simeq H\backslash G
\end{tikzcd}$$

Proposition 5.1.5 Let T be a maxl torus of H.

$H \circlearrowright V := (\mathfrak{u}/\mathfrak{u}_+)\oplus S$ ← Symp rep of H.

Define $\Xi :=$ nonzero weights of T acting on V.

(symplectic $\Leftrightarrow \Xi = -\Xi$).

$c_2 := c_2(V) \in H^4(BH, \mathbb{Z})$.

(a) M is strongly anomaly-free if $c_2(V) \equiv 0 \bmod 2$

equivalently, $\sum_{\chi \in \Xi/\{\pm\}} \chi \in 2X^*(T)$.

(b) M is anomaly-free if $\exists$ char $\theta : H \to \mathbb{G}_m$

s.t. $c_2(V) \equiv c_1(\theta)^2 \bmod 2$.

Equivalently $$\sum_{\chi \in \Xi/\{\pm 1\}} \chi \in \underbrace{X^*(T)^W}_{\underset{X^*(T)}{\wr\wr}} + 2X^*(T)$$.

<u>Proof</u> $\underbrace{H \backslash G = M_\circ}\hookrightarrow M$ is a homotopy equivalence

$\exists!$ closed $G \times \mathbb{G}_{gr}$-orbit.

$$H^4_G(M, \mathbb{Z}) \simeq H^4_G(H\backslash G, \mathbb{Z}) = H^4(BH, \mathbb{Z})$$

$$\overset{\in}{c_2(TM)} \rightsquigarrow c_2(TM|_{H\backslash G}) \longleftrightarrow H\text{-rep'n } \mathfrak{g}/\mathfrak{h} \oplus (\mathfrak{g}/\mathfrak{h})^e \oplus S.$$

Formally write (as $\mathfrak{sl}_2 \times \mathfrak{h}$-rep'n)

$$\mathfrak{g}/\mathfrak{h} = \bigoplus_m \underbrace{\mathrm{Sym}^m(\mathrm{std}_2) \otimes W_m}_{\text{some rep'ns of } \mathfrak{h}}.$$

$$(\mathfrak{g}/\mathfrak{h}) \oplus (\mathfrak{g}/\mathfrak{h})^e \oplus S \underset{\underset{\text{as } \mathfrak{h}\text{-rep'ns.}}{\uparrow}}{=} \bigoplus_m W_m^{\oplus m+2} \oplus S$$

$$\overset{\text{"mod 2"}}{\equiv} \left(\bigoplus_{m \text{ odd}} W_m\right) \oplus S.$$

When m odd $\Longleftrightarrow \mathrm{Sym}^m(\mathrm{std}_2)$ has a wt 1 subrep of dim 1.

So abstractly, as H-rep'ns,

$$\left(\bigoplus_{m \text{ odd}} W_m\right) \oplus S \simeq V.$$

For 2nd part of (a)(b):

<u>Lem</u> H reductive, $T \subseteq H$ max torus $/\mathbb{C}$. Then

(a) $[*/T] \to [*/H]$ defines an isom $H^4(BH,\mathbb{Z}) \simeq H^4(BT,\mathbb{Z})$.

(b) $\mathrm{Sym}^2 X^*(T) \xrightarrow{\sim} \mathrm{Sym}^2 H^2(BT,\mathbb{Z}) \simeq H^4(BT,\mathbb{Z})$

(c) $c_2(V)$ of Prop is equal to $\sum_{\chi\in\Xi/\{\pm1\}} -\chi^2 \in \mathrm{Sym}^2 X^*(T)$.

As a rep'n of $T\subseteq H$, $V = \bigoplus_{\chi\in\Xi}\chi$

$$c(V) = \prod_{\chi\in\Xi}(1+c_1(\chi)t)$$

$$= \prod_{\chi\in\Xi/\{\pm1\}} (1\underbrace{+c_1(\chi)t)(1-c_1(\chi)}_{=1-c_1(\chi)^2t^2}t)$$

Example If M admits a distinguished polarization,

i.e. $S = S^+ \oplus S^-$ as H-rep'ns (both max'l isotropic)

and $\eta_1 = 0$.

$\Rightarrow c_2(V) = c_1(\det S^+)^2 \mod 2$.

$\Rightarrow M$ is anomally free.

Example $G = Sp(V)$, $M = V$ not anomally free.

But it's possible for $H \leq G$, $H \curvearrowright M$ is AF.

e.g. $SO_{2n}\times Sp_{2m} \to Sp_{4mn}$, $E_7 \to Sp_{56}$, $SL_6 \xrightarrow{\wedge^3} Sp_{20}$.

$Spin_{10} \to Sp_{16}$, $Spin_{11} \to Sp_{32}$, $Spin_{12} \to Sp_{32}$.

e.g. $SL_2 \xrightarrow{\mathrm{Sym}^3} Sp_4$ not hyperspherical

(for the connectedness issues).

e.g. (Anomalous) $SO_{2n+1}\times Sp_{2m} \hookrightarrow Sp_{2m(2n+1)}$

Alternative anomalous condition [BDF+22]

M satisfies anomally condition in [BDF+22] if Prop 5.1.5(b) holds after pulling back to H_{sc}

i.e. $c_2(V)|_{BH_{sc}} \equiv 0$ in $H^4(BH_{sc}, \mathbb{Z}) \otimes \mathbb{Z}/2\mathbb{Z}$.

Prop Assume Conj 4.3.16 about (G, M).
If $P(x) = B$, then $(G^\vee, M^\vee)$ satisfies the anomaly vanishing cond. of [BDF$^+$22].

§2 Hyperspherical dual pair

Expectation There's a duality
switching anomally-free hyperspherical $(G, M) \leftrightarrow (G^\vee, M^\vee)$
s.t. when M admits a distinguished polarization
this was defined before.

§3 Hyperspherical dual pairs over Spec $\mathbb{Z}$

Expectation $\exists$ a distinguished "split" form of
each non-anomallous hyperspherical (G, M)
denoted by $(G, M)_{\mathbb{Z}}$ / Spec $\mathbb{Z}$.

$\mathcal{D}(G, M) = \{ \iota : H \hookrightarrow G$, commuting $\mathfrak{sl}_2$-pair $(h, f) \in \mathfrak{g}$
and $\rho : H \to Sp_{2g}$ s.t. h carries from a cochar $\mathbb{G}_m \to G \}$

$\updownarrow$

$\mathcal{D}_+(G, M) = \{ —, \rho^+ : H \to GL_g = GL(S^+), — \}$

$\uparrow$

$S = S^+ \oplus S^-$.

Prop'n Suppose given $(G \times \mathbb{G}_{gr}, M) / \mathbb{C}$
$\rightsquigarrow \mathcal{D}_{\mathbb{C}}$ linear alg data.

Then $\exists\, p_0, N \gg 0$ s.t. when p prime $\geq p_0$, $\mathbb{F}$ containing $\mathbb{F}_{p^N}$

$\exists$ at most one, up to isom, datum $\mathcal{D}_{\mathbb{F}}$ s.t.

$$\mathcal{D}_{\mathbb{F}} \longleftarrow \mathcal{D}_{\mathbb{Z}[\frac{1}{N_G}]} \longrightarrow \mathcal{D}_{\mathbb{C}}.$$

Moreover, if $\mathrm{Aut}\,\mathcal{D}_{\mathbb{C}}$ is connected, then may take $N=1$.

<u>Proof</u> $Z := \mathrm{Aut}(\mathcal{D})$. Then all choices of $\mathcal{D}_{\mathbb{F}_{p^k}}$

are parametrized by $H^1(\mathrm{Gal}_{\mathbb{F}_{p^k}}, Z)$.

<u>Fact</u> (Lang) $H^1(\mathbb{F}_p, Z) \longrightarrow H^1(\mathbb{F}_{p^k}, Z)$ vanishes

If $\#\pi_0(Z)\ \#\mathrm{Aut}(\pi_0(Z)) \mid k$.

$\pi_0(Z)$ is bounded uniformly.