# Shearing & Geometric Satake (I)

Xingzhu Fang

## § Shearing of vector spaces

$M = \bigoplus_{i\in\mathbb{Z}} M_i \in \mathrm{Rep}(\mathbb{G}_m)$,

$\rightsquigarrow \quad M^{\text{⧄}} = \bigoplus_{i\in\mathbb{Z}} M_i[i], \quad M^{\text{⧅}} = \bigoplus_{i\in\mathbb{Z}} M_i[-i]$.

Have $(M\otimes N)^{\text{⧄}} \cong M^{\text{⧄}} \otimes N^{\text{⧄}}$ e.g. $k^{\text{⧄}} \otimes k^{\text{⧄}} \cong (k\otimes k)^{\text{⧄}}$

$\rightsquigarrow \quad \text{⧄} \curvearrowright \mathrm{Rep}(\mathbb{G}_m)$.

Rmk 2 variants : (1) $\text{⧄} \curvearrowright \mathrm{Rep}^{\text{super}}(\mathbb{G}_m)$ (super rigid)

$M^{\text{⧄}} = \bigoplus_{i\in\mathbb{Z}} \Pi^i M[i]$, $\Pi$ = "change of parity".

$\rightsquigarrow$ symmetric monoidal str of $\mathrm{Rep}^{\text{super}}(\mathbb{G}_m)$.

$\text{⧄} : \mathrm{Rep}(\mathbb{G}_m) \xrightarrow{\sim} \mathrm{Rep}^{\text{super}}_{\varepsilon}(\mathbb{G}_m)$

i.e. parity determined by $-1 \in \mathbb{G}_m$.

(2) Frobenius : $\langle 1\rangle = [1](\tfrac{1}{2})$ (fixing $q^{1/2}$).

(3) Combining (1) & (2).

## § Categories w/ group action

$G$ alg grp ($G \curvearrowright X$)

2 choices

$$\begin{array}{ccc} (\mathrm{Rep}(G), \otimes)\text{-mod} & \ni & \mathrm{QCoh}(X/G) \\ \text{eq} \uparrow \downarrow \text{deeq} & & \text{eq} \uparrow \downarrow \text{deeq}. \\ (\mathrm{QCoh}(G), *)\text{-mod} & \ni & \mathrm{QCoh}(X) \end{array}$$

Ex Case $G$ = finite grp

$\mathrm{QCoh}(G)\text{-mod} \simeq \{\text{cats w/ } G\text{-action}\}$

$\simeq \{(\mathcal{C}, F_g \curvearrowright \mathcal{C}, \mu_{g,h} : F_g \circ F_h \to F_{gh})\}$

$$\begin{tikzcd} \mathrm{QCoh}(G)\text{-mod} \arrow[r, bend left, "\text{equivariantization}"] & (\mathrm{Rep}\, G)\text{-mod} \arrow[l, bend left, "\text{deequivariantization}"] \end{tikzcd} = \mathcal{O}(G)\text{-comod}.$$

$$\mathcal{C} \longmapsto \mathcal{C}^G := \left\{ \begin{array}{l} (X, u_g) : X \in \mathrm{Ob}(\mathcal{C}), \\ u_g : F_g(x) \xrightarrow{\sim} X \\ u_g \circ F_g(\mu_h) = \mu_{gh} \circ u_g(x) \end{array} \right\}$$

$$\underline{\mathcal{D}} = \left\{ \begin{array}{l} \mathcal{O}(G)\text{-mod in } \mathcal{D} \\ (X, \mathcal{O}(G) \circ X \to X) \\ \text{morph of } \mathcal{O}(G)\text{-mod} \end{array} \right\}^{\wedge} \longleftarrow\!\shortmid\ \mathcal{D}$$

· General situation : $G$ linear alg grp.

$$(\mathrm{QCoh}(G), *)\text{-mod} \simeq \mathrm{ShvCat}(BG)$$

$$\Gamma(*, -) \longleftarrow\!\shortmid \ \overset{\shortparallel}{} \left\{ \begin{array}{l} \Gamma(S, \mathcal{C}) \in \mathrm{QCoh}(S)\text{-mod}, \ \forall S \to BG \\ \mathrm{QCoh}(S_1) \otimes_{\mathrm{QCoh}(S_2)} \Gamma(S_2, \mathcal{C}) \xrightarrow{\sim} \Gamma(S_1, \mathcal{C}) \\ \qquad \mathrm{Spec}\, S_1 \xrightarrow{f} \mathrm{Spec}\, S_2. \end{array} \right\}$$

$$\left( G \times G \ \substack{\xrightarrow{pr_1} \\ \xrightarrow{\text{mod}} \\ \xrightarrow{pr_2}} \ G \rightrightarrows * \right) \longmapsto BG$$

Then

- $$\begin{tikzcd} \mathrm{ShvCat}(BG) \arrow[r, bend left, "eq = \Gamma(BG, -)"] & \mathrm{QCoh}(BG) \arrow[l, bend left, "deq = \Gamma(S, \mathcal{C})"] \end{tikzcd} \simeq \mathrm{Rep}\, G\text{-mod}.$$

$$\mathrm{QCoh}(S) \otimes_{\mathrm{QCoh}(BG)} \mathcal{C} \longleftarrow\!\shortmid \ \mathcal{C}$$

- $$\mathrm{QCoh}(G)\text{-mod} \longrightarrow \mathrm{Rep}\, G\text{-mod}$$

$$\mathcal{C} \longmapsto \mathcal{C}^G := \mathrm{Hom}_{\mathrm{QCoh}(G)}(\mathrm{Vect}, \mathcal{C})$$

$$\mathcal{D} \otimes_{\mathrm{Rep}G} \mathrm{Vect} \longleftarrow\!\shortmid \ \mathcal{D}$$

Ex (Cartier duality)

$$\begin{tikzcd}
(\mathrm{Rep}\,\mathbb{G}_m, \otimes)\text{-mod} \arrow[d, shift left, "\deg"] \arrow[r, phantom, "\simeq"] & (\mathrm{QCoh}(\mathbb{Z}), *)\text{-mod} \arrow[d, shift left, "\mathrm{eq}"] & \ni \mathcal{C} \\
(\mathrm{QCoh}(\mathbb{G}_m), \times)\text{-mod} \arrow[u, shift left, "\mathrm{eq}"] \arrow[r, phantom, "\simeq"] & (\mathrm{Rep}\,\mathbb{Z}, \otimes)\text{-mod} \arrow[u, shift left, "\deg"] & \arrow[u, bend right]
\end{tikzcd}$$

$\mathcal{C}$ loc sys of cats on $S^1 = B\mathbb{Z} = \bigcirc$

w/ $\mathbb{Z}$-action = monodromy

~~taking~~ global secs = eq.

- $$\begin{array}{ccc} (\mathrm{QCoh}(\mathbb{G}_m), *)\text{-mod} & \simeq & (\mathrm{Rep}\,\mathbb{G}_m, \otimes)\text{-mod} \\ \circlearrowleft & & \circlearrowleft \\ \varparallelogram & & \varparallelogram : V \otimes M^{\varparallelogram} = (V^{\varparallelogram} \otimes M)^{\varparallelogram}. \end{array}$$

<u>Examples</u> (1) Graded (dg) $k$-alg $A$

$$\begin{array}{l} \qquad\qquad\qquad\qquad\qquad \mathrm{QCoh}(X) \\ \qquad\qquad\qquad\qquad\qquad\quad \parallel \\ X = \mathrm{Spec}\,A \circlearrowleft \mathbb{G}_m \rightsquigarrow (A\text{-mod})^{\varparallelogram} \simeq A^{\varparallelogram}\text{-mod} \\ \qquad\qquad\qquad\qquad\qquad (A\text{-mod}^{gr})^{\varparallelogram} \simeq A^{\varparallelogram}\text{-mod}^{gr} \\ \qquad\qquad\qquad\qquad\qquad\quad \parallel \\ \qquad\qquad\qquad\qquad\qquad \mathrm{QCoh}(X/\mathbb{G}_m). \end{array}$$

$X^{\varparallelogram} = \mathrm{Spec}\,A^{\varparallelogram}$ but $\mathrm{QCoh}(X^{\varparallelogram}) \neq \mathrm{QCoh}^{\varparallelogram}(X)$.

(2) Twisting $G$-rep : $z : G \to \mathbb{Z} \rightsquigarrow z : BG \to B\mathbb{Z}$

$$G_F \xrightarrow{z} \mathbb{G}_{m,F} \xrightarrow{\mathrm{val}} \mathbb{Z}.$$

$z^* : (\mathrm{QCoh}(\mathbb{G}_m), *) = (\mathrm{Rep}(\mathbb{Z}), \otimes) \longrightarrow (\mathrm{SHV}(BG), \otimes) \longrightarrow Z(\mathrm{SHV}(G))$

$\mathrm{ShvCat}(BG) = \mathrm{SHV}(G)$-mod. linear over $\mathrm{SHV}(BG)$.

Shear $\mathrm{SHV}(G)$-mod $\mathcal{C} \mapsto \mathcal{C}^{\varparallelogram}$.

(3) Shear $G$-rep'n.

- $\varpi : \mathbb{G}_m \to \mathrm{Aut}(G) \rightsquigarrow \mathbb{G}_m \times BG \to BG$

$$\rightsquigarrow \mathrm{Rep}\,G \in \mathbb{G}_m^{\mathrm{mod}}$$

with $\mathrm{Rep}(G)^{\underline{\tilde{\varpi}}\mathbb{Z}} = k[G]^{\underline{\tilde{\varpi}}\,\varparallelogram}$-comod.

- $\mathrm{Rep}\,G \rightsquigarrow \mathrm{Rep}(G \rtimes \mathbb{G}_m)$

· $\tilde{\varpi} : \mathbb{G}_m \to G$, $\mathrm{Rep}\,G \simeq \mathrm{Rep}^{\tilde{\varpi}\square}$ or $G \rtimes \mathbb{G}_m \xrightarrow{\sim} G \times \mathbb{G}_m$

$$(g, t) \longmapsto (g\tilde{\varpi}(t), t)$$

· $G \circlearrowright X$, $\mathrm{QCoh}(X/G)^{\square} \simeq \mathrm{QCoh}(X/G)$

$\mathrm{QCoh}(X)^{\square} \simeq \mathrm{QCoh}(X)$.

(4) Koszul duality.

$$\xrightarrow{\quad \underbrace{\text{derived}}_{\text{connective}} \quad 0 \quad \underbrace{\text{stacky}}_{\text{coconnective}} \quad}$$

· $R$ conn $\rightsquigarrow$ $R\text{-mod} = \mathrm{QCoh}(R)$.

· $A$ coconn $\rightsquigarrow$ $X = \mathrm{Spec}\,A$, $\mathrm{QCoh}(X) = \varprojlim_{\substack{\mathrm{Spec}R \to \mathrm{Spec}A \\ R \text{ coconn}}} \mathrm{QCoh}(\mathrm{Spec}\,R)$

$\rightsquigarrow$ $\mathrm{QCoh}(X^{\square}) \neq \mathrm{QCoh}^{\square}(X) = A^{\square}\text{-mod}$

· $A = k[x_2]$, $x_2$ deg 0 & $\mathbb{G}_m$-wt $= -2$,

$B = k[y_{-1}]$, $y_{-1}$ coh deg $-1$ & $\mathbb{G}_m$-wt $= 2$,

$$(-)\otimes_{A^{\square}} k : A^{\square}\text{-mod} \rightleftarrows B\text{-mod} : \mathrm{Hom}_B(k, -)$$

$$H^*_{\mathbb{G}_m}(\mathbb{G}_m) = k \longmapsto (A^{\square}[-2] \xrightarrow{x_2} A^{\square}) \otimes_{A^{\square}} k = B[-1]$$

$$H^*_{\mathbb{G}_m}(pt) = A^{\square} \longleftarrow k$$

$$\left(\begin{array}{ccc} A^{\square} = H^*_{\mathbb{G}_m}(pt) & & H_*(\mathbb{G}_m) = B \\ \circlearrowright & & \circlearrowright \\ H^*_{\mathbb{G}_m}(X) & \hookleftarrow & H^*(X) \end{array}\right)$$

$$k[x_2, x_2^{-1}] \longmapsto \mathrm{Per}(\cdots \to k[y_{-1}] \xrightarrow{y_{-1}} k[y_{-1}] \to \cdots) \neq 0$$

in $\mathrm{IndCoh}(B) \supset \mathrm{QCoh}(B)$.

<u>Thm</u> (Koszul duality)

$$\begin{tikzcd} \mathrm{QCoh}^{\square}(\mathbb{A}^1) \arrow[r, phantom, "\simeq"] & \mathrm{IndCoh}(\mathbb{A}^1[-1]) \arrow[d, bend right, "\Psi"'] \\ \mathrm{QCoh}((\mathbb{A}^1)^{\square}) \arrow[u, hook] \arrow[r, phantom, "\simeq"] & \mathrm{QCoh}(\mathbb{A}^1[-1]) \arrow[u, hook] \end{tikzcd}$$

· $\mathrm{Perf} \overset{\mathrm{Ind}}{\hookrightarrow} \mathrm{QCoh} \underset{\Psi}{\overset{\Xi}{\rightleftarrows}} \underset{\underset{\mathcal{F}}{\Psi}}{\mathrm{IndCoh}} \overset{\mathrm{Ind}}{\hookleftarrow} \mathrm{Coh}$, $\mathrm{QCoh} \simeq \mathrm{IndCoh}_{\{0\}}$.

- $k \in \mathrm{Coh}(\mathbb{A}^1[-1])$, $\quad 0 \to k \to k[y_{-1}] \to k[y_{-1}] \to \cdots$

  deg $\quad 1 \quad 3 \quad 5 \quad \cdots$

  $KD(k) = A^{\square}$,

  $\Xi(k) = (\cdots \to k[y_{-1}] \to k[y_{-1}] \to 0) = KD(\Xi(k))$

  $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \overset{\shortparallel}{A^{\square}[x_2^{-1}] / A^{\square}[-1]}$.

(5) Abelian geometric Satake.

$$\mathrm{Perv}(L^+G \backslash Gr_G) \cong (\mathrm{Rep}\,\check{G})^{\heartsuit}$$

Recall its proof: highest wt str $\longrightarrow$ Schubert cells

wt str $\longrightarrow$ semi-infinite orbits $= H^*(-) = \bigoplus$ orbits.

$\otimes$ str $\longrightarrow$ Convolution product.

$\rightsquigarrow \mathrm{Perv}(L^+G \backslash Gr_G) \simeq \square((\mathrm{Rep}\,\check{G})^{\heartsuit}) \subset (\mathrm{Rep}\,\check{G})^{\square}$

(6) Derived geom Satake.

$$\begin{array}{ccccc} \mathrm{SHV}(L^+G \backslash Gr_G) & \simeq & \mathrm{QCoh}^{\square}(\check{\mathfrak{g}}^* / \check{G}) & \simeq & \mathrm{IndCoh}(\check{\mathfrak{g}}^*[-1] / \check{G}) \\ \uparrow & & \uparrow & & \uparrow \\ \mathrm{SHV}_s(L^+G \backslash Gr_G) & \simeq & \mathrm{QCoh}(\check{\mathfrak{g}}^{*\square} / \check{G}) & \simeq & \mathrm{IndCoh}_{\check{N}}(\check{\mathfrak{g}}^*[-1] / \check{G}) \end{array}$$

Heuristic construction of derived Satake transform

**Step 1** $\quad \mathrm{Rep}\,\check{G} \simeq D((\mathrm{Rep}\,\check{G})^{\heartsuit}) \simeq D(\mathrm{Sat}_G) \longrightarrow \mathrm{Sph}_G \xrightarrow{\mathrm{Av}} \mathrm{Whit}_G := \mathrm{SHV}(Gr_G)^{(LN, \psi)}$

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad \overset{\shortparallel}{\mathrm{Sph}_G^{\heartsuit}}$

$\rightsquigarrow$ geometric Casselman-Shalika $\mathrm{Rep}\,\check{G} \simeq \mathrm{Whit}_G$.

**Step 2** $\quad$ Factorize Step 1.

**Step 3** $\quad \mathrm{Sph}_G \xrightarrow{*} Z_{\mathbb{E}_2}(\mathrm{Whit}_G) = Z_{\mathbb{E}_2}(\mathrm{Rep}\,\check{G}) = \mathrm{QCoh}((B\check{G})^{S^2})$

$\qquad = \mathrm{QCoh}((pt \times_{\check{\mathfrak{g}}} pt) / \check{G}) = \mathrm{QCoh}(\check{\mathfrak{g}}^*[-1] / \check{G})$.

**Step 4** $\quad$ Normalize.