Shearing & Geometric Satake (II)

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$G / \bar{\mathbb{F}}_p[[t]]$ split reductive grp

t ($\mathbb{C}[[t]]$ or $W(\bar{\mathbb{F}}_p)$)

Define L^+G loop group scheme rep'd by the functor

$$\mathrm{Alg}_{\bar{\mathbb{F}}_p} \longrightarrow \text{Groupoid}$$
$$R \longmapsto G(R[[t]])$$

LG ind-scheme : $LG(R) = G(R((t)))$, $R((t)) = R[[t]][\frac{1}{t}]$

$Gr_G := LG/L^+G$ ind-scheme / $\bar{\mathbb{F}}_p$.

L^+G $= \bigcup_{\lambda \in X_*(T)} \overline{Gr_\lambda}$ Schubert varieties

$\overline{Gr_\lambda} = \overline{L^+G \cdot \lambda(t) \cdot L^+G / L^+G}$.

finite type proj var / $\bar{\mathbb{F}}_p$.

<u>Thm</u> (Geometric Satake, Mirkovic-Vilonen, 07)

$$(\mathrm{Perv}(L^+G \backslash Gr_G, \bar{\mathbb{Q}}_\ell), *) \cong (\mathrm{Rep}(\hat{G}), \otimes)$$

$*$ = convolution product $\hat{G}$: Langlands dual grp / $\bar{\mathbb{Q}}_\ell$.

equiv of <u>symm monoidal categories</u>.

Convolution product:

$$\begin{tikzcd}
L^+G\backslash LG \times^{L^+G} LG/L^+G \arrow[r, "m"] \arrow[d, "p"'] \arrow[rd, "q"] & L^+G\backslash LG/L^+G \\
L^+G\backslash LG/L^+G & L^+G\backslash LG/L^+G
\end{tikzcd}$$

$$\begin{tikzcd}
(x,y) \arrow[r, mapsto] \arrow[d, mapsto] \arrow[rd, mapsto] & xy \\
x & y
\end{tikzcd}$$

(m proper) $\Downarrow$ $m_* = m_!$

$\mathcal{F}, \mathcal{G} \in \mathrm{Perv}(L^+G \backslash \mathrm{Gr}_G)$

$\rightsquigarrow \mathcal{F} * \mathcal{G} := m_*(p^*\mathcal{F} \otimes q^*\mathcal{G})[\cdots]$

where $LG \times_{L^+G} LG = LG \times LG / \sim$

with $(x, gy) \sim (gx, y)$ for $g \in L^+G$.

§ Symmetric monoidal categories

Set

monoidal set:

set $S + m : S \times S \longrightarrow S$

$+ p : \{*\} \longrightarrow S$

satisfying associativity

+ identity

(properties of m and p)

Cat

monoidal cat

cat $\mathcal{C}$ + functors $m : \mathcal{C} \times \mathcal{C} \longrightarrow \mathcal{C}$

$+ \; 1 \xrightarrow{e} \mathcal{C}$, 1 = triv cat

satisfying associativity

+ identity

$$\begin{tikzcd} \mathcal{C} \times \mathcal{C} \times \mathcal{C} \arrow[r, "m \times id"] \arrow[d, "id \times m"'] & \mathcal{C} \times \mathcal{C} \arrow[d, "m"] \\ \mathcal{C} \times \mathcal{C} \arrow[r, "m"] \arrow[ur, phantom, "\overset{=}{\alpha}"] & \mathcal{C} . \end{tikzcd}$$

i.e. natural isom $\alpha : m \circ (m \times id) \cong m \circ (id \circ m)$

identity : natural isom $m \circ (id \times p) \cong id \cong m \circ (p \times id)$

(extra structure : $\alpha, \rho_1, \rho_2 \cdots$)

Symmetric monoidal : property of monoidal S

$$\begin{tikzcd} \text{i.e.} \quad S \times S \arrow[r, "m"] \arrow[dr, bend right] & S \\ & S \times S / (x,y) \sim (y,x) \arrow[u] \end{tikzcd}$$

braided (symmetric) monoidal cat :

monoidal cat $(\mathcal{C}, m, p, \alpha, \rho_1, \rho_2)$

+ natural isom $\gamma : m \cong m \circ sw$

i.e. $m(X, Y) \underset{\gamma_{X,Y}}{\cong} m(Y, X)$ natural in X & Y.

(Symmetric : γ has the extra property $\gamma_{X,Y} \circ \gamma_{Y,X} = id$.)

called commutative constraints.

<u>§ Commutativity constraints of geometric Satake</u>

- On the dual grp side $Rep(\hat{G})$: $V \otimes W \xrightarrow[\gamma]{\sim} W \otimes V$ canonical.
- On the geometric side $Perv(L^+G \backslash Gr_G)$:

  Constraints given by fusion product (highly nontrivial)

<u>Beilinson-Drinfeld affine Grassmanian</u>

$$Gr_{X^2} \longrightarrow X^2, \quad X = \mathbb{A}^1 = \operatorname{Spec} \overline{\mathbb{F}}_p[t]$$

(or $Gr_{X^n} \longrightarrow X^n$.)

parametrizing $\{(x,y) \in X^2, \mathcal{G}$ G-torsor over X, $\mathcal{G}|_{X \setminus \{x,y\}} \cong G|_{X \setminus \{x,y\}}\}$

isom of G-torsors.

$$\begin{tikzcd}
Gr_{X^2}|_{X^2\setminus\Delta} \arrow[d, hook, "j"] & \cong (Gr_X \times \underbrace{Gr_X})|_{X^2\setminus\Delta} \arrow[d, "s\ (\text{special for } X = \mathbb{A}^1)"] \\
Gr_{X^2} & Gr_G \times X \arrow[d, "p"] \\
 & Gr_G \\
Gr_{X^2}|_{\Delta} \cong Gr_X = Gr_G \times X \arrow[uu, hook, "i"] & Gr_G \times \{*\} \arrow[l, hook', "d"']
\end{tikzcd}$$

<u>Definition</u> The fusion product is

$$\mathcal{F} \tilde{\otimes} \mathcal{G} := d^* i^* j_{!*}(p^*\mathcal{F} \boxtimes q^*\mathcal{G}|_{X^2\setminus\Delta})[-]$$

where $j_{!*} : Perv(\mathcal{U}) \longrightarrow Perv(X)$, $\mathcal{U} \subset X$ open.

The fusion product is the same as convolution product on Perv side.

$$\mathcal{F} \tilde{\otimes} \mathcal{G} \cong \mathcal{F} * \mathcal{G}.$$

fusion product is symmetric,

$$\exists \ \mathcal{F} \tilde{\otimes} \mathcal{G} \simeq \mathcal{G} \tilde{\otimes} \mathcal{F}$$

(coming from symmetry of $\boxtimes$).

$$\begin{array}{ccc} \mathrm{Perv}^{\text{ULA}/X}_{L^+G}(\mathrm{Gr}_X) & \mathrm{Gr}_G \times X & \\ \text{\rotatebox{90}{$\cong$}} & \wr & \\ \mathrm{Perv}(\mathrm{Gr}_G) & \mathrm{Gr}_G & \end{array}$$

$$\begin{tikzcd} \mathrm{Gr}_X \arrow[d] & \mathrm{Gr}_G \arrow[l, hook'] \arrow[d] \\ X & * \arrow[l] \end{tikzcd}$$

<u>Problem</u> The natural functor $\mathrm{Perv}(L^+G \backslash \mathrm{Gr}_G, \bar{\mathbb{Q}}_\ell) \longrightarrow \mathrm{Rep}(\hat{G})$

is induced from the cohomology functor $\bigoplus_i H^i(\mathrm{Gr}_G, -)$

which is only graded commutative,

the commutativity constraits of Perv is <u>NOT</u> compatible

with the commutativity constraints of $\mathrm{Rep}(\hat{G})$.

<u>Solution</u> (1) Modify the constraints of $\mathrm{Perv}(L^+G \backslash \mathrm{Gr}_G)$

by hand by a sign.

(2) Change the cat $\mathrm{Rep}(\hat{G})$.

define an action of $\mathbb{G}_m$ on $\bar{\mathbb{Q}}_\ell[\hat{G}]$ (ring of fcns on $\hat{G}$)

by $\mathrm{Ad} \circ (2\rho)$, $2\rho = \sum_{\lambda \in \Delta_+} \lambda$,

i.e. $t \cdot f(g) = f(2\rho(t)^{-1} g \cdot 2\rho(t))$.

$\rightsquigarrow$ shear $\bar{\mathbb{Q}}_\ell[\hat{G}]^{\sharp} := \bigoplus_{n \in \mathbb{Z}} \underset{\{tf = t^n f\}}{\underset{\parallel}{\bar{\mathbb{Q}}_\ell[\hat{G}]_n}} [n]$

abelian cat $\qquad$ Hopf dga with trivial differential.

$$\begin{array}{l} \mathrm{Rep}_{2\rho}(\hat{G})^{\sharp} := \left\{ \begin{array}{l} \text{graded } \bar{\mathbb{Q}}_\ell[\hat{G}]^{\sharp}\text{-mods. i.e. complexes of f.d. v.s. } V \text{ with} \\ \text{comod str } V_* \to V_* \otimes \bar{\mathbb{Q}}_\ell[\hat{G}]^{\sharp} \text{ s.t. } V_* = (W^{2\rho})^{\sharp} \text{ for } W \in \underset{\text{(abelian cat)}}{\mathrm{Rep}(\hat{G})} \end{array} \right\} \\ \cap \\ \mathrm{CoMod}(\bar{\mathbb{Q}}_\ell[\hat{G}]^{\sharp}) \end{array}$$

where $(W^{2\rho})^{\square} = \bigoplus W_n[n]$

$\phantom{where (W^{2\rho})^{\square} = \bigoplus} \overset{\shortparallel}{\{w \in W \mid 2\rho(t)w = t^n w,\ \forall t \in \mathbb{G}_m\}}$

$\mathrm{Rep}_{2\rho}(\hat{G})^{\square}$ inherits commutativity constraints.

<u>Formulate the geometric Satake</u> (abelian ver)

$$\mathrm{Perv}_{L^+G}(L^+G \backslash Gr_G, \bar{\mathbb{Q}}_\ell) \simeq \mathrm{Rep}_{2\rho}(\hat{G})^{\square}.$$

between symmetric monoidal cats

§ Derived geometric Satake

<u>Thm</u> (Bezrukavnikov - Finkelberg, 2007)

$$(D_{cons}(L^+G \backslash Gr_G, \bar{\mathbb{Q}}_\ell), *) \simeq \mathrm{Perf}(\mathrm{Sym}^*(\hat{\mathfrak{g}}^*[2])/\hat{G})$$

(A_∞-cat) $\qquad$ symmetric monoidal.

$\shortparallel$

as monoidal dg cats (∞-cat that are module cat w.r.t. E_∞-monoidal ∞-cat $D(\mathrm{Mod}_{\bar{\mathbb{Q}}_\ell})$.)

Compatible with Frobenius, with Frob action on Perf stably defined using shearing.

Set	Cat	∞-Cat
monoid	monoidal cat	A_1-cat, $\cdots$, A_∞-cat $= E_1$-cat
Symm monoid	braided	E_2-cat
	symmetric	E_3-cat, $\cdots$, E_∞-cat

Symm monoidal ∞-cat.

<u>Thm</u> (Nocera, 2023)

$D_{cons}(L^+G \backslash Gr_G, \bar{\mathbb{Q}}_\ell)$ is an E_3-cat (in the topological setting $G/\mathbb{C}[\![t]\!]$.)

<u>Open question</u> Does derived Satake lift to an equiv of E_3-cats?

<u>Partial answer</u>

<u>Thm</u> (Campbell - Raskin, 2023)

The derived Satake lifts to an equiv of functorizable cats ($\rightsquigarrow E_2$-cats)