The local conjecture (I)

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Plan
- Introduction to local conj.
- Difficulties with sheaf theory.
- Nadler-Gaitsgory's result on spherical orbits
  - $\rightsquigarrow$ Knop, Brion (1980's)
- Combinatorics of spherical varieties.

(G, M) split spherical pair $/\mathbb{F}$ ($= \mathbb{C}$ or $\bar{\mathbb{F}}_q$).

M dist polarization, $M \simeq T^*(X, \Psi)$

$\rightsquigarrow (G^\vee, M^\vee)$ dual pair $/\mathbb{C}$ or $\bar{\mathbb{Q}}_\ell$.

$F = \mathbb{F}((t))$, $\mathcal{O} = \mathbb{F}[[t]]$

$X_\mathcal{O}$ rep'ting $R \mapsto X(R[[t]])$.

X_F rep'ting $R \mapsto X(R((t)))$.

<u>Local conjecture</u> $SHV(X_F/G_\mathcal{O}) \simeq QC^\shortparallel(M^\vee/G^\vee)$

basic obj $\quad \delta_X \longleftrightarrow \mathcal{O}^\shortparallel_{M^\vee/G^\vee}$

Hecke action on $X_F/G_\mathcal{O} \longleftrightarrow$ Hecke action on $M^\vee/G^\vee$

via derived Satake.

Frob $\circlearrowright X_F \longleftrightarrow \mathbb{G}_{gr} \circlearrowright M^\vee$

loop rotation $\longleftrightarrow$ Poisson str

factorization $\longleftrightarrow$ factorization

Smooth affine $M^\vee \circlearrowleft G^\vee$ Hamiltonian.

<u>Input</u> $\mu: M^\vee \to \mathfrak{g}^{*\vee}$ $\mathbb{G}_{gr}$-equiv.

$\mathcal{O}^{\square}_{M^\vee}$ cotangent bundle.

$M^\vee$ vec bun / $G^\vee/G^\vee_x$ with fiber V_x

$$QC^{\square}(M^\vee/G^\vee) \simeq QC^{\square}(V_x/G^\vee_x).$$

$\mu: M^\vee/G^\vee \to \mathfrak{g}^{\vee *}/G^\vee$

Get Hecke action from μ: one side of der Satake.

$$QC^{\square}(M^\vee/G^\vee) \times QC^{\square}(\mathfrak{g}^{*\vee}/G^\vee) \to QC^{\square}(M^\vee/G^\vee)$$

$$(\mathcal{F}, \mathcal{G}) \longmapsto (\mathcal{F} \otimes \mu^* \mathcal{G})$$

Compatible w/ tensors on $QC^{\square}(\mathfrak{g}^{*\vee}/G^\vee)$

$QC^{\square}(M^\vee/G^\vee)$ is a module cat / derived Hecke cat.

$M^\vee$ affine $\Rightarrow$ $M^\vee/G^\vee$ affine over $\mathfrak{g}^{*\vee}/G^\vee$.

$QC^{\square}(M^\vee/G^\vee)$ generated as module cat by $\mu_* \mathcal{O}_{QC^{\square}(\mathfrak{g}^{*\vee}/G^\vee)}$.

<u>Computation of Hom</u>

$V \in Rep(\check{G}) \rightsquigarrow \underline{V} = \mathcal{O}_{M^\vee} \otimes V$ $\quad G^\vee \times \mathbb{G}_{gr}$ equiv sheaf on $M^\vee$.

<u>Affiness</u> $\underline{V}$ generated by $QC^{\square}(M^\vee/G^\vee)$

$$Hom(\underline{V}, \underline{W}) = Hom^{G^\vee}(Hom(V, W), \mathcal{O}^{\square}_{M^\vee}).$$

Hecke sheaves $\mathcal{H}^{\times}_G$ = $G_{\mathcal{O}}$-equiv shv on X_F

$\bar{\mathcal{H}}^{\times}_G$ = $G_{\mathcal{O}}$-equiv SHV on X_F.

§ Difficulties with $SHV(X_F/G_O)$

X_O represents $R \mapsto X(R[\![t]\!])$

$\Rightarrow \quad X_O = \lim X(R[t]/t^{n+1}) = \lim X_n$.

X affine $\Rightarrow$ X_n affine $\Rightarrow$ X_O affine.

X smooth $\Rightarrow$ X_n smooth $\Rightarrow$ X_O pro-smooth.

Fix G-equiv embedding $X \hookrightarrow V$.

X^l = pts of X_F in $t^{-l} V[\![t]\!]$.

X^l_n = pts of X_F in $t^{-l} V[\![t]\!] / t^{n+1} V[\![t]\!]$

with $X_F = \operatorname{colim}_l \lim_n X^l_n$.

Rmk We have little control on $X^l_{n+1} \to X^l_n$ (even if + nice input on X).

e.g. $X = (Q(x,y,z) = x^2 + y^2 + z^2 = 1)$.

$X^1_0 = X_F(tV[\![t]\!]/t) = ((x_{-1}t^{-1} + x_0)^2 + (y_{-1}t^{-1} + y_0)^2 + (z_{-1}t^{-1} + z_0)^2 = 1)$.

$$\underset{\substack{\| \\ 0}}{(x_{-1}^2 + y_{-1}^2 + z_{-1}^2)} t^{-2} + \underset{\substack{\| \\ 0}}{(2x_{-1}x_0 + 2y_{-1}y_0 + 2z_{-1}z_0)} t^{-1} + \underset{\substack{\| \\ 1}}{(x_0^2 + y_0^2 + z_0^2)}$$

Say $V_{-1} = (x_{-1}, y_{-1}, z_{-1})$, $V_0 = (x_0, y_0, z_0)$

$\Rightarrow \quad Q(V_{-1}) = 0, \ V_{-1} \perp V_0, \ Q(V_0) = 1. \qquad X^1_1 \to X^1_0$

Consider fibres of $X^1_1 \to X_0$:

over V_{-1}, fibre is nonzero

over V_1, fibre $= 0$ $\quad (Q(V_0) = 1, \ V_1 \perp V_0)$.

Phenomenon Hard to control fibres (with jumping dims).

Thm (Drinfeld, Ngô: Grinberg–Kazhdan formal arc thm).

X sch, f.t. $/k$, $X^\circ \subset X$ smooth.

$L(X)$ formal arcs, $L^\circ(X)$ arcs not contained in $X \setminus X^\circ$.

Fix $\gamma_0 : \operatorname{Spec} k[[t]] \to X$ in $L^\circ X$.

$\exists\, Y$ finite type,

$$(LX)^\wedge_{\gamma_0} \simeq \hat{Y}_y \times D^\infty \simeq \hat{Y}_y \times \prod \operatorname{Spf} k[[t]].$$

Raskin 2017 (unpublished)

$$\mathrm{SHV}^!(X_F) = \underset{X \to \mathcal{U} \text{ f.t.}/k}{\operatorname{colim}} \mathrm{SHV}(\mathcal{U})$$

$\uparrow$ $\mathcal{U}_1 \xrightarrow{f} \mathcal{U}_2$ with $!$-pullback compatibility.

$f : X \to Y$, $f^! : \mathrm{SHV}^!(Y) \to \mathrm{SHV}^!(X)$

$\Delta : X \to X \times X \rightsquigarrow \otimes^!$

If $f : X \to Y$ is ind-proper, $f_* \dashv f^!$.

But No Verdier duality.

Also, $\mathrm{SHV}^*(X) = \underset{\substack{X \to \mathcal{U} \\ \text{f.t.}/k}}{(\mathrm{co})\lim} \mathrm{SHV}(\mathcal{U})$ with $\mathcal{U}_1 \xrightarrow{p} \mathcal{U}_2$, p_* pushforward,

"dual sheaf theory".

Upshot If $\mathrm{SHV}^!(X)$ is dualizable, then its dual is $\mathrm{SHV}^*(X)$.

$\Rightarrow$ Some form of Verdier duality.

Def (Placid rep) $X = \lim \mathcal{U}_i$ filtered inverse lim of finite type schs,

s.t. $\mathcal{U}_{i+1} \to \mathcal{U}_i$ smooth affine translation maps.

For smooth maps $f : \mathcal{U}_{i+1} \to \mathcal{U}_i$, $f^* \dashv f_*$, $f^![\dim f] \dashv f_*$.

Fact If X is placid, $\exists$ canonical $\omega_X^{ren} \in \mathrm{Shv}^*(X)$

If X prosm, $\exists$ canonical $\omega_X^{ren} = p^* k$ for $p : X \to pt$.

Always, $SHV^!(X) \circlearrowleft SHV^*(X)$

X placid $\Rightarrow$ action induces equiv $W_X \simeq W_X^{ren}$

$$SHV^!(X) \simeq SHV^*(X).$$

<u>Placid+</u> $X_F = \operatorname{colim} X^\ell$, X^ℓ G_O-stable,

$X^\ell = \lim_n X^\ell_n$, X^ℓ_n f.t.

& each $X^\ell_{n+1} \to X^\ell_n$ torsors for abel unipotent grp.

$$\begin{array}{ccc} G_O \times X^\ell_n & \longrightarrow & X^\ell_n \\ & \searrow \quad \nearrow & \\ & G_N \times X^\ell_n & \end{array}$$

Assume X_F placid.

- Verdier duality $\mathbb{D}$.

  Basic obj $\delta_X \rightsquigarrow \mathbb{D}\delta_X$

  $\delta_X = \pi_\square k$, $\pi : X_O \to X_F$.

- $T \in D(Gr_G)$, $\mathbb{D}(T * \mathcal{F}) = \mathbb{D}(T) * \mathbb{D}(\mathcal{F})$.

- $Shv^\ell_n = Shv\, X^\ell_n / G_N$, $Shv^\ell_n \to Shv(X_F/G_O)$

  $X^\ell_n/G_N \leftarrow X^\ell_n/G_O \leftarrow X^\ell/G_O \to X_F/G_O$.

  Can take $Shv^\ell_n \to Shv^\ell_{n+1}$ & $Shv^\ell_n \hookrightarrow Shv^{\ell+1}_n$

  $(X^\ell_{n+1}, G_N) \to (X^\ell_n, G_N)$ $(X^\ell_n, G_N) \leftarrow (X^\ell_n, G_{N''}) \to (X^\ell_n, G_{N'})$.

Easy to define

$\mathrm{Hom}_{Shv(X_F/G_O)}(\iota_1 \mathcal{F}, \iota_2 \mathcal{G})$ $(\iota_i : Shv^{\ell_i}_{n_i} \to Shv(X_F/G_O))$.

$= \operatorname{holim}_K \operatorname{holim}_N \mathrm{Hom}_{X^K_N}(\mathcal{F}, \mathcal{G})$.

§ Gaitsgory–Nadler (+ Knop, Brion, Luna, Vust)

TFAE:
- G/S spherical. $(S \subseteq G)$
- $S(\mathcal{K})$ acts on Gr_G with countable orbits.
- $G(\mathcal{O})$ acts on $(G/S)(\mathcal{K})$ with countable orbits.

Always, X smooth & spherical. $\overset{\circ}{X} \simeq G/S$.

Tori compactification

$\Lambda_X^{\vee,+} \subset \Lambda_+$ weights s.t. $\mathbb{C}[X]_\lambda$ nonzero.

$A = \mathrm{Spec}(\mathbb{C}[\Lambda_X^\vee])$, Λ_A dual to $\Lambda_X^\vee$. $\Lambda_X^{pos} \subset \Lambda_A$.

$\rightsquigarrow \quad \bar{A} = \mathrm{Spec}(\mathbb{C}[\Lambda_X^{\vee,+}])$.

$G(\mathcal{O})$ orbits on $\overset{\circ}{X}(\mathcal{K})$ will be a subset of Λ_A.

$$G(\mathcal{O}) \text{ orbits on } (G/S)(\mathcal{K}) = X(\mathcal{K}) \setminus (X \setminus \overset{\circ}{X})(\mathcal{K}) / G(\mathcal{O})$$

$$\downarrow \nu$$

$$\bar{A}(\mathcal{K}) \setminus (\bar{A} \setminus A)(\mathcal{K}) / A_{\mathcal{O}} \simeq \Lambda_A$$

① $\mathcal{V}(G/S) \xleftrightarrow{\text{bij}} \left(\text{finitely generated sub semigrp of full rank in } \Lambda_A\right)$.

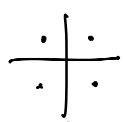
$\quad \| $

G-inv val'ns on G/S

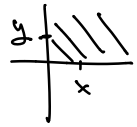
$= G(\mathcal{O})$-orbits on $\overset{\circ}{X}(\mathcal{K})$

② A $G(\mathcal{O})$-orbit $\mathbb{O}$ is contained in $X(\mathcal{O})$ iff $\nu(\mathbb{O}) \in \Lambda_X^{pos}$.

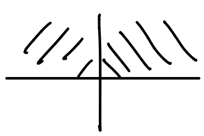
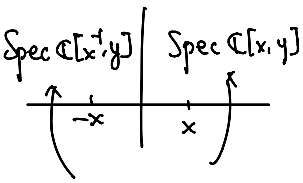
Toric examples Toric variety is given by a fan

(1)  $\rightsquigarrow$ x^{-1}, x, y^{-1}, y

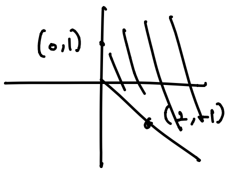
triv fan $\mathrm{Spec}\, \mathbb{C}[x, x^{-1}, y, y^{-1}] = \mathbb{P}^2$.

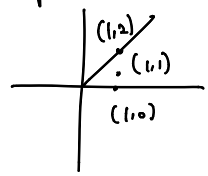
(2) 

$\rightsquigarrow$ $\operatorname{Spec} \mathbb{C}[x,y] = \mathbb{A}^2$.

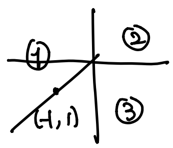
(3)  $\rightsquigarrow$ 

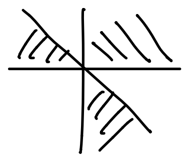
$\operatorname{Spec} \mathbb{C}[x,x^{-1},y] = \mathbb{P}^1 \times \mathbb{A}^1$

(4) 

$\rightsquigarrow$ 

$\operatorname{Spec} \mathbb{C}[x, xy, xy^2]$
$\quad \|$
$\operatorname{Spec} \mathbb{C}[u,v,w]/(v^2-uw)$.

(5) 

$\rightsquigarrow$  get $\mathbb{P}^2$.

Key affine $\Leftrightarrow$ fan is cone
smooth $\Leftrightarrow$ generators of dual red grp are part of a basis.
fan contains whole lattice $\Leftrightarrow$ space is complete

Def A spherical var is simple if it contains a unique closed G-orbit.

Lem Any sph var admits a covering by simple sph vars.

X sph, $\mathring{X} \simeq G/S$,

$\mathbb{C}(X)^{(B)} = \{ f \in \mathbb{C}(X) \mid bf = \chi(b) f \}$ for some χ.

$\Lambda(X) = \{ \chi_f \mid f \in \mathbb{C}(X)^{(B)} \}$.

Consider val'ns on $\mathbb{C}(X)$.

Divisors $D \longmapsto \nu_D$

$\rho_X : \{\text{discrete val'ns on } X\} \longrightarrow N(X) = \mathrm{Hom}(\Lambda(X), \mathbb{Q})$.

$\mathcal{V}(X)$ = G-invariant discrete val'ns.

<u>Thm</u> $\rho_X|_{\mathcal{V}(X)} : \mathcal{V}(X) \longrightarrow N(X)$ is injective.

<u>Classification of simple embeddings</u>

<u>Def</u> A colored cone in $N(G/H)$ is $(\mathcal{C}, \mathcal{D})$

- $\mathcal{C} \subset N(X)$, $\mathcal{D} \subset \Delta(X)$

  (Δ colors: B-invariant but not G-invariant discrete vals).
- $\mathcal{C}$ strictly convex poly cone generated by

  $\rho_X(\mathcal{D})$ and finite subset of $\mathcal{V}(G/H)$.

  $\not\ni 0$

<u>Example</u> $G = SL_2$, $H = T$, $SL_2/T = (\mathbb{P}^1 \times \mathbb{P}^1) \setminus \Delta(\mathbb{P}^1)$.

B-stable subset $([x:1], [y:1])$ $x \neq y$

$(x-y)^{-1} \in \mathbb{C}(SL_2/T)$, B-semi-inv of wt α_1

$f(x,y) = x - y$, $\Lambda(G/H) = \mathbb{Z}\alpha_1$.

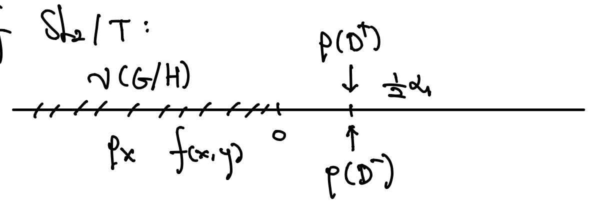
Colors are $D^+ = \mathbb{P}^1 \times \{[1:0]\}$

$D^- = \{[1:0]\} \times \mathbb{P}^1$

$f(x,y)$ ($\mapsto -\alpha_1$) has poles of order 1 along D^+ & D^-.

$\langle \rho(D^\pm), -\alpha_1 \rangle = -1$.

Picture of SL_2/T:



2 Color cones: $(\{0\}, \emptyset) \rightsquigarrow SL_2/T$ triv embedding

$(\mathcal{V}(G/H), \emptyset) \rightsquigarrow \mathbb{P}^1 \times \mathbb{P}^1$