The local conjecture (II)

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<u>Last time</u> Difficulties w/ sheaf theory (§7.1 – §7.3).

<u>Today</u> · 2 Thms of Gaitsgory – Nadler + Spherical examples.

· odds & ends.

For $S \subset G$, TFAE:

- G/S spherical.
- $S(K)$ acts on Gr_G with countable orbits.
- $G(\mathcal{O})$ acts on $(G/S)(K)$ with countable orbits.

[GN] $\Lambda_X^{\vee,+} \subset \Lambda_T^{\vee}$ dom wts s.t. $\mathbb{C}[X]_\lambda \neq 0$, X (Smooth) spherical var.

$A = \operatorname{Spec} \mathbb{C}[\Lambda_X^\vee]$, Λ_A dual of $\Lambda_X^\vee$,

$\bar{A} = \operatorname{Spec} \mathbb{C}[\Lambda_X^{\vee,+}]$, X° open in X that is isom to G/S.

Identify $G(\mathcal{O})$-orbits on $X^\circ(K)$ as a subset of Λ_A.

$$\rightsquigarrow \begin{tikzcd} (G(\mathcal{O})\text{-orbits on } (G/S)(K)) \arrow[d, "\simeq"] \arrow[dd, bend left, "\nu"] \\ X(K) \backslash (X \backslash \mathring{X})(K) / G(\mathcal{O}) \arrow[d] \\ (\bar{A}(K) \backslash (\bar{A} \backslash A)(K) / A(\mathcal{O})) \simeq \Lambda_A \end{tikzcd}$$

(1) $\mathcal{V}(G/S) \xleftrightarrow{\text{bij}}$ fin gen saturated semisubgrp of full rank in Λ_A.

(2) A $G(\mathcal{O})$-orbit $\mathbb{O}$ is contained in $X(\mathcal{O})$

$$\iff v(\mathbb{O}) \subset \Lambda_X^{+} \ (= \Lambda_X^{pos}).$$

Let X sph var, $X^\circ = G/S$

$\rightsquigarrow \Lambda(X) = \{\chi_f \mid f \in \mathbb{C}(X)^{(B)} \text{ with char } \chi\}$

$\mathrm{Hom}(\Lambda(X), \mathcal{O}) = N(X)$.

<u>Key</u> Classification of

$\{$simple embeddings of $G/S\}$

$\updownarrow$ Last time.

$\{$Colored cones $(\mathcal{C}, \mathcal{D})\}$

<u>Ex 1</u> $G = SL_2$, $S = T$, $SL_2/T \simeq (\mathbb{P}^1 \times \mathbb{P}^1) \setminus \Delta(\mathbb{P}^1)$.

B = stable subset, $([x:1],[y:1]) \subset \mathbb{P}^1 \times \mathbb{P}^1 \setminus \Delta(\mathbb{P}^1)$.

$(x-y)^{-1} \in \mathbb{C}(G/S)^{(B)}$ of wt α_1, $f := x-y$.

$\rightsquigarrow \Lambda(SL_2/T) \simeq \mathbb{Z}\alpha_1$.

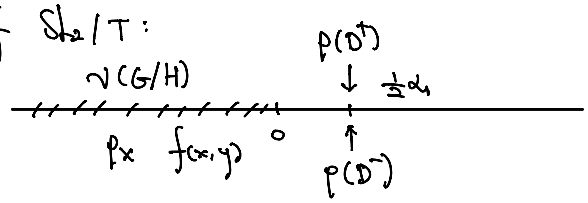
$N(SL_2/T) \simeq \mathbb{Z}\, \omega_1^\vee$.

Colors are $D^+ = \mathbb{P}^1 \times \{[1:0]\}$ & $D^- = \{[1:0]\} \times \mathbb{P}^1$

$f(x,y)$ $(\mapsto -\alpha_1)$ has poles of order 1 along D^+ & D^-.

$\langle \rho(D^\pm), -\alpha_1 \rangle = -1$.

Picture of SL_2/T:



2 color cones: $(\{0\}, \emptyset) \rightsquigarrow SL_2/T$ triv embedding

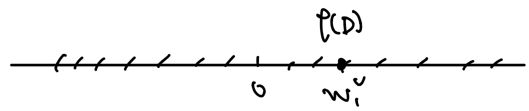
$(v(G/H), \emptyset) \rightsquigarrow \mathbb{P}^1 \times \mathbb{P}^1$

Ex 2 $G = SL_2$, $S = \mathcal{U}$, $SL_2/\mathcal{U} \simeq \mathbb{C}^2 \setminus \{(0,0)\}$.

The unique color is $\{y=0\} \subseteq SL_2/\mathcal{U}$.

The function y is B-semi-inv with char α_1 (or ω_1 ?)

$\mathcal{V}(G/S) = N(X)$, $N(G/S) = \mathcal{V}(G/S)$.



Then the compactifications are

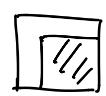
$$(0, \emptyset) \longleftrightarrow SL_2/\mathcal{U} = \mathbb{C}^2 \setminus \{0,0\}$$
$$(\mathbb{Q}_{\le 0}\, \rho(D), \emptyset) \longleftrightarrow SL_2/\mathcal{U} \cup (\text{line at } \infty)$$
$$(\mathbb{Q}_{\ge 0}\, \rho(D), D) \longleftrightarrow \mathbb{C}^2$$
$$(\mathbb{Q}_{\ge 0}\, \rho(D), \emptyset) \longleftrightarrow Bl_{(0,0)}\, \mathbb{C}^2 \qquad \text{"max cpt'n".}$$

Ex 3 $G = SL_3$, $S = SO_3 \cdot Z(SL_3)$

$\Rightarrow$ G/S = space of sm conics in $\mathbb{P}^2$.

$SO_3 = \{A^{-1} = {}^tA\}^\circ$,

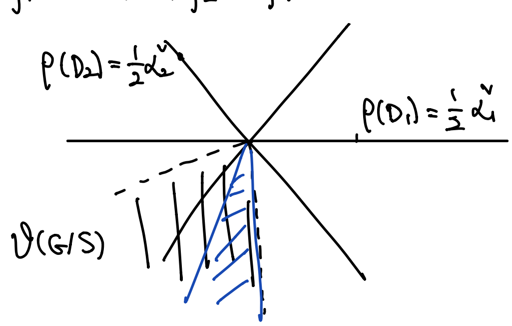
f_1 = lower right 2×2-matrix of $A \cdot {}^tA$



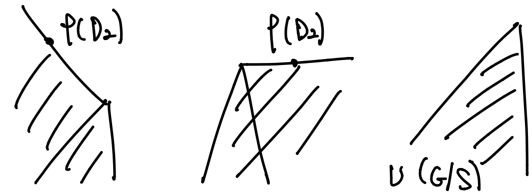
f_2 = lower right entry of $A \cdot {}^tA$.

Then f_1/f_2^2, f_2/f_1^2 B-semi-inv on G/S of weights $2\alpha_1, 2\alpha_2$, basis of $\Lambda(G/S)$.

Colors are $\{f_1 = 0\}$ & $\{f_2 = 0\}$.



e.g.



(wonderful cpt'n)

Generalize from colored cones:

- Any spherical var $\xleftrightarrow{\text{bij}}$ colored fan in $N(G/S)$.
- X is complete $\Leftrightarrow$ $\mathcal{V}(G/S) \subset \bigcup \mathcal{C}$

<u>Note</u> Arbitrary sph var $X \xleftrightarrow{\text{bij}}$ colored fans [Krop91]

Call X <u>toroidal</u> if none of the cones are colored.

<u>Functoriality</u> S_1 another sph subgrp $\phi: G/S \to G/S_1$.

X, X_1 resp toroidal cpt'ns + $\Phi: X \to X_1$.

$\rightsquigarrow$ $\phi_*: \mathcal{N}(G/S) \to \mathcal{N}(G/S_1)$

$\phi: \mathcal{V}(G/S) \to \mathcal{V}(G/S_1)$.

This ϕ extends to $\hat{\Phi}: X \to X_1$

$\Leftrightarrow$ ϕ_* maps $\mathcal{F}$ to $\mathcal{F}_1$

When $\mathcal{V}(G/S)$ strictly convex,

$\exists!$ wonderful cpt'n X of G/S.

$\mathcal{C} = \mathcal{V}(G/S)$, color $(\mathcal{C}, \phi)$.

<u>Thm</u> (Brion, Polo) TFAE:

- $\exists$ simple complete toroidal cpt'n of G/S (wonderful).
- $N_G(S)/S$ finite
- $\mathcal{V}(G/S)$ strictly convex.

Let S does not satisfy these, $S_1 = S \cdot N_G(S)^\circ$

$\rightsquigarrow G/S \longrightarrow \underbrace{G/S_1}_{\text{has wonderful cptn.}}$

<u>Setups of [GN]</u>

$X^\circ \subset X$, B^{op} s.t. $B^{op}S$ open in G,

P^{op} s.t. $PB^{op}S = B^{op}S$,

M Levi subgrp of P^{op}. U^{op} unipotent radical.

$A = M/M \cap S \longrightarrow G/S$,

$\mathbb{C}(G/S)^{B^{op}} \longrightarrow \mathbb{C}(A)$, $\quad N(G/S) \simeq \Lambda_A$

$\Lambda^+_{G/S} \subset \Lambda$ given by $\mathcal{V}(G/S)$.

Y^+ toric var corresp to $\Lambda^+_{G/S}$ in Λ_A.

[BLV] $U^{op} \times Y^+ \longrightarrow X$ open embedding

which contains open nonempty subset of

each G-orbit in X.

Via $A \to G/S$, identify $\mathcal{V}(G/S)$ with subset Λ_A,

A, $\Lambda^+_{G/S}$ as subsets of $(G/S)(K)$.

<u>Thm</u> Each elt of $(G/S)(K)$ contains a unique elt of $\Lambda^+_{G/S}$

(1) (Ex) $G = H \times H$, $S = \Delta H$

$\Rightarrow$ Cartan decomp $H(\mathcal{O}) \backslash H(K) / H(\mathcal{O}) \simeq \Lambda^+_H$

(2) (Ex) $S = U$, $\quad G(\mathcal{O}) \backslash G(K) / U(K) \simeq \Lambda_G$.

Proof of [GN]:

Let X = any complete toroidal cpt'n of G/S.

$$(G/S)(K) \hookrightarrow X(K).$$

X complete $\Rightarrow \gamma \in X(K)$ extends to $\bar{\gamma} \in X(\mathcal{O})$

$$U^{op} \times Y^{+} \longrightarrow X \text{ open orbit}$$

$\rightsquigarrow$ $G(\mathcal{O})$-orb that $\bar{\gamma}$ contains $\bar{\gamma}'$,

$$\bar{\gamma}' \in U^{op}(\mathcal{O}) \times Y^{+}(\mathcal{O})$$

Act by $U^{op}(\mathcal{O}) \rightsquigarrow \bar{\gamma}'' \in Y^{+}(\mathcal{O})$

Also act by $T(\mathcal{O}) \rightsquigarrow \bar{\gamma}'''$ corresp to $\lambda \in \Lambda^{+}_{G/S}$.

Now take $\lambda, \lambda' \in \Lambda^{+}_{G/S}$ distinct.

Can show they are not in the same $G(\mathcal{O})$-orbit.

$$\lambda, \lambda' : \mathbb{G}_m \hookrightarrow G/S \hookrightarrow X.$$

If $S(K)$ has countable orbits on $G/S \Rightarrow S$ sph.

λ dom reg cowt, $G \cdot t^{\lambda} \simeq G/B$.

Here $S(K)$ has countable orbits on G/B

$\exists\, \mu$ s.t. $S(K)\mu \cap G\cdot t^{\lambda}$ is dense in $G/B = G\cdot t^{\lambda}$.

Take $\operatorname{Lie} S \twoheadrightarrow \mathfrak{g}/\mathfrak{b}$

$\Rightarrow S$ has an open orbit in G/B.

<u>Local Conj</u> $\quad SHV(X_F/G_{\mathcal{O}}) \simeq QC^{\#}(M^{\vee}/G^{\vee})$.

- δ_X = strict sheaf of $X_{\mathcal{O}} \longleftrightarrow \mathcal{O}^{\#}_{M^{\vee}/G^{\vee}}$.
- Hecke action

$$SHV(G_{\mathcal{O}}\backslash X_F/G_{\mathcal{O}}) \xleftrightarrow{\text{Derived Satake}} (QC^{\#}(\check{\mathfrak{g}}^{*}/G^{\vee}), \otimes)$$

- $\text{Frob} \curvearrowright X_F \longleftrightarrow \mathbb{G}_{gr} \curvearrowright M^{\vee}$

- Loop rot $\mathbb{G}_m^{rot}$ on X_F $\longleftrightarrow$ Poisson str on $M^\vee$
- "Factorization" $\longleftrightarrow$ Factorization.

$M = \mathbb{A}^1$, $G = \mathbb{G}_m$, $(\mathbb{A}^1)^\ell = t^{-\ell} \cdot \mathbb{R}[\![t]\!]$.

$$\underline{K}_{X^\ell} \longrightarrow \underline{K}_{X^{\ell+1}}$$

$$\underline{K}_{X^\ell} \longrightarrow \underline{K}_{X^{\ell+1}}\langle 2\rangle \quad (\text{from } W_{X^{\ell+1}} \to W_{X^\ell}).$$

$$\operatorname{colim}(\underline{K}_{X^\ell} \to \underline{K}_{X^{\ell+1}}) = \delta_0 \in SHV(\mathbb{A}^1/G_0)$$

$$\operatorname{colim}(\underline{K}_{X^\ell} \to \underline{K}_{X^{\ell+1}}\langle 2\rangle) = \mu \in SHV(\mathbb{A}^1/G_0)$$

$$[\operatorname{Hom}(\mathcal{F}, \delta_0)^\vee] = f(0)\, \mu(X(\mathcal{O})/G(\mathcal{O}))$$

$$[\operatorname{Hom}(\mathcal{F}, \mu)^\vee] = \int_{X(F)/G(\mathcal{O})} f(x)\, dx.$$

<u>Ex</u> $M = M_n$, $G = GL_n \times GL_n$

T_j Hecke sheaf for $GL_n \longleftrightarrow \Lambda^j \mathrm{Std}^*$.

$$T_j * \delta_X = \underline{I}_j \langle j(n-j)\rangle,$$

$\underline{I}_j$ is GL_n^2-equiv sheaf supp'd on $M_{n,\mathcal{O}}$ whose fibre at $S \in M_{n,\mathcal{O}}$ is cohom of j-dim subspaces $E \subset k^n$ s.t. $S|_E = \{0\}$.

X admits nowhere vanishing eigenchar $\eta : G \to \mathbb{G}_m$, $X = S^+ \times^{H_u} G$ with $\deg \eta : G_F \to \mathbb{G}_{m,F} \to \mathbb{Z}$.

$$SHV^{\# \deg \eta}(X_F/G_0) \simeq QC^{\#, \text{neut}}(M^\vee/G^\vee)$$

$\rightsquigarrow$ Shearing both sides:

$$SHV(X_F/G_0) \simeq QC^{\#, \text{neut}, \eta^\vee, \partial p}(M^\vee/G^\vee)$$

<u>Rmk</u> In $SHV^{\# \deg}(M_n/GL_n^2)$, $T_j * \delta_X = \underline{I}_j\langle -j^2\rangle$.

<u>WWLi thinks</u>: the punchline here is the independence of n for some autom convenience in practice.

<u>L^2-theory of sph vars, Arthur SL_2, ...</u>

$L^2(X)$ should consist to simple Arthur type.

- $SL_2 \to G$, $\mu: M^\vee \to \mathfrak{g}^{*\vee}$

$f \mapsto \mathfrak{g}^\vee$ $\rightsquigarrow \mathcal{N}_{<f}$.

Local conj $\Rightarrow$ $SHV(X_F/G_O)$ anti-tempered cat annihitated by $O^{\Pi}_{N_{<f}} \in QC^{\Pi}(\mathfrak{g}^{*\vee}/G^\vee)$.

$\mathbb{G}_m^{rot} \curvearrowright X_F, X_O, G_O, \cdots$

$\mathbb{G}_m^{rot} \curvearrowright SHV(X_F/G_O)$

$\rightsquigarrow SHV(X_F/G_O)^{\mathbb{G}_m^{rot}}$ mod out $H^r_{\mathbb{G}_m^{rot}}(pt, k) \simeq k[u]$, u coh deg 2.

Have a full subcat $SHV(X_F/G_O) \otimes_{k[u]} k \hookrightarrow SHV(X_F/G_O)$

Local conj $\Rightarrow$ This is equiv to $SHV(X_F/G_O)^{\mathbb{G}_m^{rot}}$ of u-deformation.

<u>note</u> $[W] \in HH^*(QC^{\Pi}(M^\vee/G^\vee))$

$\mapsto$ defines a class $HH^*(SHV(X_F/G_O))$.

e.g. $X = H$, $G = H \times H$ + derived Satake

$X = pt$, $G = G$ + derived Satake recovers many properties.