Planchrel algebra

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k alg closed field, char $k = 0$ or $k = \bar{\mathbb{F}}_q$.

$F = k((t)) \supseteq \mathcal{O} = k[[t]]$.

X/k spherical var of G, G/k reductive.

T^*X hyperspherical.

Take $M^\vee = V_X \times^{G_X^\vee} G^\vee$ affine sch.

<u>Local conj</u>

$$SHV(X_F/G_\mathcal{O}) \simeq QC^\square(M^\vee/G^\vee) \ni \mathcal{O}(M^\vee)^\square$$

$$SHV(G_F/G_\mathcal{O}) \simeq QC^\square(\mathfrak{g}^{*\vee}/G^\vee) \quad \text{str sheaf of } M^\vee$$

period Satake.

<u>Baby case</u> (non-derived) $\mathrm{Rep}(\check{G}) \circlearrowright \mathcal{C}$

Inner Hom: $\mathcal{F}, \mathcal{G} \rightsquigarrow \mathrm{Hom}_\mathcal{C}(V \otimes \mathcal{F}, W \otimes \mathcal{G})$

$$\parallel$$

$$\mathrm{Hom}_{\mathrm{Rep}(\check{G})}(V, W \otimes \underline{\mathrm{Hom}}(\mathcal{F}, \mathcal{G}))$$

where $V, W \in \mathrm{Rep}(\check{G})$.

i.e. $\underline{\mathrm{Hom}}(\mathcal{F}, -) : \mathcal{C} \to \mathrm{Rep}(\check{G})$ is right adjoint of

$$\mathrm{Rep}(\check{G}) \to \mathcal{C}, \quad V \mapsto V \otimes \mathcal{F}$$

Same senario works for ∞-cats (under mild assumption)

$\rightsquigarrow$ can define (for $\mathcal{H}_G \circlearrowright QC^\square(M^\vee/\check{G})$)

$$\underline{\mathrm{End}}(\mathcal{O}(M^\vee)^\square) = \mathcal{O}(M^\vee)^\square \in \mathrm{Alg}(\mathcal{H}_G)$$

$$\mathcal{H}_G = QC(\mathfrak{g}^{*\vee}/G^\vee)^\square$$

By local conj. $\underline{\mathrm{End}}(\mathcal{O}(M^\vee)^\square) \longleftrightarrow \underline{\mathrm{End}}(\delta_X) =: \mathbb{PL}^X$
where $\delta_X = i_* \mathbb{W}$, $i: X_\mathcal{O} \to X_F$.

Conj We have an isom b/w DG-algebras over $\mathcal{O}(\mathfrak{g}^{*\vee})^\square$ with $\check{G}$-action

$$\underset{\substack{\uparrow \\ \text{Plancherel alg}}}{\mathbb{PL}_X} \simeq \mathcal{O}(M^\vee)^\square$$

also compatible with Frob- & G_{gr}-actions when $k = \overline{\mathbb{F}_q}$.

$$\mathbb{PL}_X = \bigoplus V \otimes \mathbb{PL}_X^{(V)}.$$

Relative Grassmannian

From now on, X is a quasi-affine sm G-var.

Def The $\underline{\text{relative Grassmannian}}$ Gr^X is def'd s.t.

$$Gr^X(\mathrm{Spec}\,R) := \{(P, \varphi, s)\}$$

where
- P is G-bundle over $\mathrm{Spec}\,R \times D$ ($D = \mathrm{Spec}\,k[\![t]\!]$, $D^* = \mathrm{Spec}\,k((t))$).
- $\varphi : P|_{\mathrm{Spec}R \times D^*} \xrightarrow{\sim} G \times (\mathrm{Spec}R \times D^*)$.
- s a section of $P \overset{G}{\times} X$ defined over $\mathrm{Spec}\,R \times D$ s.t. $\varphi(s)$ is def'd on $\mathrm{Spec}\,R \times D$.

Roughly. Gr^X classifies (x, g), $x \in X_\mathcal{O}$, $g \in Gr_G$ s.t. $xg \in X_\mathcal{O}$.

$$g \longleftrightarrow (P, \varphi), \quad x \longleftrightarrow \varphi(S)$$

$$\Rightarrow i^{Gr} : Gr^X \longrightarrow X_\mathcal{O} \times Gr_G.$$

Claim i^{Gr} gives a locally closed embedding for Gr^X to $X_\mathcal{O} \times Gr_G$.

$\rightsquigarrow$ enough to treat X affine.

Choose equivariant embedding $X \hookrightarrow V$ (V vec space)

$$G \to GL(V)$$

$\rightsquigarrow$ can assume X v.s. & $G = GL_n$.

Then an R pt of $X_{\mathcal{O}} \times Gr_G$

$$\updownarrow \ 1\text{-}1$$

$x \in R[[t]]^n$, $M \subseteq R((t))^n$

s.t. M is locally free, $M[t^{-1}] = R((t))^n$.

Note $(x, M) \in Gr^X \iff x \in M$ $\rightsquigarrow$ closed condition.

$G_{\mathcal{O}} \curvearrowright Gr^X$ by $(x,g)k = (xk, k^{-1}g)$.

Define
$$\pi : Gr^X \longrightarrow Gr_G, \quad (x,g) \longmapsto g,$$
$$p_1 : Gr^X \longrightarrow X_{\mathcal{O}}, \quad (x,g) \longmapsto x,$$
$$p_2 : Gr^X \longrightarrow X_{\mathcal{O}}, \quad (x,g) \longmapsto xg.$$

Write
$$Gr^X_{F,\mathcal{O}} := X_{\mathcal{O}} \times^{G_{\mathcal{O}}} G_F,$$
$$Gr^X_F := X_F \times^{G_{\mathcal{O}}} G_F.$$

By abuse of notation, $\pi : Gr^X \to G_{\mathcal{O}} \backslash G_F / G_{\mathcal{O}}$
$$\tilde{\pi} : Gr^X_{F,\mathcal{O}} \to G_{\mathcal{O}} \backslash G_F / G_{\mathcal{O}},$$
$$\pi_F : Gr^X_F \longrightarrow G_{\mathcal{O}} \backslash G_F / G_{\mathcal{O}}.$$

Then
$$\begin{tikzcd}
X_{\mathcal{O}}/G_{\mathcal{O}} \arrow[d, "i"'] & Gr^X/G_{\mathcal{O}} \arrow[l, "p_2"'] \arrow[r, "p_1"] \arrow[d, "i^{Gr}"'] & X_{\mathcal{O}}/G_{\mathcal{O}} \arrow[d, "i"] \\
 & X_{\mathcal{O}} \times^{G_{\mathcal{O}}} G_F/G_{\mathcal{O}} \arrow[dl, "\tilde{p}_2"'] \arrow[ur, "\tilde{p}_1"] \arrow[d, "\tilde{i}"] & \\
X_F/G_{\mathcal{O}} & X_F \times^{G_{\mathcal{O}}} G_F/G_{\mathcal{O}} \arrow[l, "p_2^F"'] \arrow[r, "p_1^F"] & X_F/G_{\mathcal{O}}
\end{tikzcd}$$

<u>Claim</u> $\mathbb{PL}_x^\vee = \mathcal{H}om_{Gr^\times_{F,\infty}}(\tilde{\Gamma}^! \mathcal{T}_v, \; i_*^{Gr} w_{Gr^\times})$.

<u>Baby case in mind</u> $\mathrm{Rep}(\check{G}) \circlearrowright \mathcal{C} \ni \mathcal{F}$.

$$\mathrm{Hom}_{\mathrm{Rep}(\check{G})}(V, \underline{\mathrm{End}}(\mathcal{F})) \longrightarrow \mathbb{PL}_x^\vee$$

$$\overset{\shortparallel}{\mathrm{Hom}(V \otimes \mathcal{F}, \mathcal{F})} \longrightarrow \mathrm{Hom}(\mathcal{T}_V * \delta_x, \delta_x).$$

<u>"Pf" of claim</u>

$$\begin{aligned}
\mathbb{PL}_x^{(V)} &= \mathcal{H}om_{X_F}((p_2^F)_* (\Gamma_F^! \mathcal{T}_v \overset{!}{\otimes} (p_1^F)^! \delta_x, \delta_x) \qquad (\delta_x = i_* w).\\
&= \mathcal{H}om_{Gr_F^\times}(\Gamma_F^! \mathcal{T}_v \overset{!}{\otimes} (p_1^F)^! \delta_x, (p_2^F)^! \delta_x)\\
&\qquad\qquad (\text{ind-properness of } p_2^F)\\
&= \mathcal{H}om_{Gr_F^\times}(\Gamma_F^! \mathcal{T}_v \overset{!}{\otimes} \tilde{i}_* w_{Gr^\times_{F,\infty}}, (p_2^F)^! \delta_x)\\
&\qquad\qquad (\text{base change})\\
&= \mathcal{H}om_{Gr_F^\times}(\tilde{i}_*(\tilde{\Gamma}^! \mathcal{T}_v \overset{!}{\otimes} w_{Gr^\times_{F,\infty}}), (p_2^F)^! \delta_x)\\
&\qquad\qquad (\text{projection formula})\\
&= \mathcal{H}om_{Gr_F^\times}(\tilde{i}_* \tilde{\Gamma}^! \mathcal{T}_v, (p_2^F)^! \delta_x)\\
&\qquad\qquad (\text{unity})\\
&= \mathcal{H}om_{Gr^\times_{F,\infty}}(\tilde{\Gamma}^! \mathcal{T}_v, \tilde{p}_2^! i_* w)\\
&\qquad\qquad (\text{adjunction})\\
&= \mathcal{H}om_{Gr^\times_{F,\infty}}(\tilde{\Gamma}^! \mathcal{T}_v, i_*^{Gr} w_{Gr^\times})\\
&\qquad\qquad (\text{base change}) \qquad \square
\end{aligned}$$

<u>Finite dim reduction</u>

Let $Gr_{G,\le n}$ be the closure of sufficiently large strata of Gr_G s.t. $\mathrm{Supp}\, \mathcal{T}_v \subseteq Gr_{G,\le n}$.

$\Gamma : X_{\mathcal{O}} \times^{G_{\mathcal{O}}} G_F \longrightarrow Gr_G$, $\quad X_N = X_{\mathcal{O}}/t^N$, $\quad G_N = G_{\mathcal{O}}/t^N$.

truncate $\quad \Gamma_N : X_N \times^{G_N} Gr_{G,\le n} \to Gr_{G,\le n}$ for $N \gg 0$.

$$i_N^{Gr} : Gr^{X}_{\le n, N} \longrightarrow X_N \overset{G_N}{\times} Gr_{G,\le n}. \qquad (Gr^X \hookrightarrow X_{\mathcal{O}} \times Gr_G)$$

<u>Assumption</u> X is placid, i.e. $X = \operatorname{colim}_n \lim_\ell X_n^\ell$.

$$\Rightarrow \quad \mathrm{Hom}_{Gr^X_{F,\mathcal{O}}}(\tilde{\Gamma}^! T_V,\ i^{Gr}_* \omega_{Gr^X})$$
$$\simeq \mathrm{Hom}_{X_N \times^{G_N} Gr_{G,\le n}}(\tilde{\Gamma}_N^! T_V,\ i^{Gr}_{N,*} \omega_{Gr^X_{\le n,N}})$$
$$\simeq \mathrm{Hom}_{Gr^X_{\le n,N}}(\underbrace{i_N^{Gr,*} \tilde{\Gamma}_N^* T_V}_{T_V^X},\ \omega_{Gr^X}).$$

<u>Conclusion</u> Pass to cohomology, we have

$$\mathbb{PL}_X^{(V)} \simeq H^*_{G_N}(\mathbb{P}T_V^X)\langle -2\dim X_N\rangle.$$

<u>Example</u> (1) $G = \mathbb{G}_m$, $X = pt$ $\Rightarrow$ $Gr^X = Gr_G$,

$$\mathbb{PL}_X = H_*^{G(\mathcal{O})}(Gr_G) = \bigoplus_n H_*^{G(\mathcal{O})}(\{z^n\})$$
$$= \bigoplus_n \mathbb{C}[w]\cdot r_n.$$

where $r_n * r_m = r_{n+m}$.

$$\mathrm{Spec}(\mathbb{C}[w, r_1, r_{-1}]/\{r_1\cdot r_{-1} = 1\}) = \mathbb{A}^1 \times \mathbb{G}_m.$$

(2) $G = \mathbb{G}_m$, $X = \mathbb{A}^1$. Then

$$\begin{tikzcd}
Gr^X = \left\{ (s, t^n) \;\middle|\; \begin{array}{l} s \in k[\![t]\!], \text{ s.t.} \\ s\cdot t^n \in k[\![t]\!] \end{array} \right\} \arrow[d] \\
X_{\mathcal{O}} \times Gr_G = \{(s, t^n) \mid s \in k[\![t]\!]\} \arrow[d] \\
Gr_G.
\end{tikzcd}
\qquad
\begin{tikzcd}
t^2\mathcal{O} \arrow[d, hook] & t^1\mathcal{O} \arrow[d, hook] & \mathcal{O} \arrow[d, hook] & \mathcal{O} \arrow[d, hook] & \mathcal{O} \arrow[d, hook] & \cdots \\
\mathcal{O} & \mathcal{O} & \mathcal{O} & \mathcal{O} & \mathcal{O}
\end{tikzcd}$$

$$\rightsquigarrow \quad H_*^{G_{\mathcal{O}}}(Gr^X) \xrightarrow{i_*} H_*^{G_{\mathcal{O}}}(X_{\mathcal{O}} \times Gr_G) \xrightarrow{z^*} H_*^{G(\mathcal{O})}(Gr_G).$$

Claim $\mathcal{Z}^*$ is an isom and $i_*\mathcal{Z}^*$ are algebra homs

i_* is injective.

Note that $w, r_i' \in H_i^{G_\mathcal{O}}(\mathrm{Gr}^X)$.

Claim $$i_*(r_i') = \begin{cases} r_i, & i \geq 0, \\ w^{-i} r_i, & i < 0. \end{cases}$$

$$\rightsquigarrow \quad \operatorname{Spec} \mathbb{C}[w, r_1', r_{-1}'] / (r_1' \cdot r_{-1}' = w) \simeq \mathbb{A}^2 = T^*\mathbb{A}^1.$$

(3) $G = PGL_2^3$, $H = \Delta PGL_2$, $X = H \backslash G$.

Preparation Gr^X when X is homogeneous classifies the data (P, φ, s)

$$\mathrm{Gr}^X / G(\mathcal{O}) = \{(P_1, P_2, \varphi, s_1, s_2)\}$$

$$\updownarrow$$

$$(P_i^H, \varphi^H) = H_\mathcal{O} \backslash H_F / H_\mathcal{O}$$

where P_i are G-bundles,

$\varphi : P_1|_{D^*} \xrightarrow{\sim} P_2|_{D^*}$

s_i sections of $P_i \times^G X$ over D s.t. $\varphi(s_1) = s_2$.

Then in (3) $\quad \mathbb{P}\mathbb{L}_X^\vee = H^*_{H(\mathcal{O})}(\mathbb{D} i^* \mathcal{T}_V)$, with $i : \mathrm{Gr}_H \hookrightarrow \mathrm{Gr}_G$.

$V = V_a \otimes V_b \otimes V_c \in \mathrm{Irr}(SL_2^3)$, $\quad \pi_0(\mathrm{Gr}_H) = \mathbb{Z}/2\mathbb{Z}$.

$\Rightarrow$ $i^* \mathcal{T}_V = 0$ unless a, b, c have the same parity.

When this happens, $i^*(\mathcal{T}_V) = k\langle a+b+c \rangle$

on each relevant strata.

(?) $\quad H^*(BH, H^*(i^* \mathcal{T}_V))$ degenerates

$\Rightarrow$ $\mathbb{P}\mathbb{L}_X^\vee \simeq H^*(\mathrm{Gr}_{H, \leq m}) \otimes H^*(BPGL_2)$.

<u>Trace of Frob</u> $\quad \mathbb{PL}_x^{(v)} \simeq \mathrm{Hom}_{Gr^{x}_{\Sigma n,N}}(\mathcal{T}_v^x, w)$.

<u>Lem</u> $k = \mathbb{F}_q$. $\mathcal{F}, \mathcal{G}$ Weil sheaves on $X/\mathbb{F}_q$

f trace func of $\mathcal{F}$

$\check{g}$ trace func of $\mathbb{D}\mathcal{G}$.

Then $\sum_{s \in X(\mathbb{F}_q)} f(s) \cdot \check{g}(s) = \mathrm{tr}(\mathrm{Frob}, \mathrm{Hom}(\mathcal{F}, \mathbb{D}\mathcal{G})^\vee)$

also valid for a weighted version

where $f(s) \cdot \check{g}(s)$ has weight $1/\# G_x(\mathbb{F}_q)$. $x \in [(X/G)(\mathbb{F}_q)]$.

Apply lemma to $\mathcal{F} = \mathcal{T}_v^x$

$\Rightarrow$ trace of Frob on $(\mathbb{PL}_x^{(v)})^\vee$ equals

$$\sum_{(x,g) \in Gr^{x}_{\Sigma n,N}} \underbrace{q^{\dim(G_N)}}_{\text{diff dual}} \cdot \frac{\overset{\text{trace}}{T_v(g)} \cdot \overset{\text{shifted action}}{\sqrt{\eta(g)}}}{\underbrace{q^{\dim(X_N)} \cdot \# G_N(\mathbb{F}_q)}_{\text{tr of } w}}$$

(from $\dfrac{q^{\dim G}}{\# G(\mathbb{F}_q)} \cdot \int_{X_0 \times G_F / G_0} T_v(g) \cdot \sqrt{\eta(g)} \cdot 1_c(xg)\, dg$

with $m(G_0) = 1$, $m(X_0) = \# X(\mathbb{F}_q)/q^{\dim X}$.)