Harder-Narasimhan stratification in p-adic Hodge theory

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(Joint with Jilong Tong).

§1 Motivation

$C :=$ Complete alg closed non-arch field / $\mathbb{Q}_p$.

$\mathcal{O}_C \rightsquigarrow k$ res field.

$\{p\text{-div gps}/k\} \xrightarrow[\sim]{\text{Dieudonné}} \{\text{Dieudonné mods}\} \xrightarrow[\text{classes}]{\text{isog}} \{\text{isocrystals}\} \to \{\text{Newton polygons}\}$

$\uparrow$ (from the bottom row to the top row)

$\{p\text{-div gps}/\mathcal{O}_C\} \xrightarrow[\sim]{\text{Scholze-Weinstein}} \{(T, W)\} \overset{?}{\dashrightarrow} \{\text{Newton polygons}\}$

T: fin free $\mathbb{Z}_p$-mod, $W \subseteq T \otimes_{\mathbb{Z}_p} C$ sub-$\mathbb{C}$-v.s.

Fix a trivialization $T \cong \mathbb{Z}_p^n$, $\dim_{\mathbb{C}} W = d$.

(?) $\mathrm{Grass}(n,d)(C) \longrightarrow \{\text{Newton polygons}\}$.

- The fiber of (?) gives a stratification on $\mathrm{Grass}(n,d)(C)$

  $\rightsquigarrow$ called the Newton stratification.

<u>Goal of the talk</u> To study an alg approximation of the Newton stratification, called HN stratification on $\mathrm{Grass}(n,d)(C)$ (and eventually on B_{dR}-Grassmannian.) } and study their relations.

§2 HN-stratification on Grass(n,d)

(By Dat-Orlik-Rapoport). $\breve{\mathbb{Q}}_p = \widehat{\mathbb{Q}_p^{ur}} \circlearrowleft \sigma$ (Frob)

$\mathrm{Fil\,Isoc}_{C/\breve{\mathbb{Q}}_p} = \{(V, \mathrm{Fil}^{\bullet} V_C)\}$,

where $\cdot\ V = \mathrm{Isoc}/\breve{\mathbb{Q}}_p$ (i.e. $V/\breve{\mathbb{Q}}_p$ fin-dim'l v.s. & $\varphi: V \xrightarrow{\sim} V$ σ-linear)

· $\mathrm{Fil}^\cdot V_C$: $\mathbb{Z}$-descending fil'n on $V_C = V \otimes_{\breve{\mathbb{Q}}_p} C$.

For $(V, \mathrm{Fil}^\cdot V_C)$: $\mathrm{rk}(V, \mathrm{Fil}^\cdot V_C) := \dim_{\breve{\mathbb{Q}}_p} V$

$$\deg(V, \mathrm{Fil}^\cdot V_C) := \deg(\mathrm{Fil}^\cdot V_C) - \underline{\dim V}$$

$$\underline{\dim V} = v_p(\det\varphi)$$

$\rightsquigarrow$ Slope = rk / deg.

Can have HN stratification.

In particular, $\forall (V, \mathrm{Fil} V_C) \in \mathrm{Fil}\,\mathrm{Isoc}_{C/\breve{\mathbb{Q}}_p}$,

Can associate a HN-vec, determined by the convex hull of

$$\{(\mathrm{rk}\, V', \deg V') \in \mathbb{R}^2 \mid (V', \mathrm{Fil}^\cdot V'_C) \text{ subobj of } (V, \mathrm{Fil}\, V_C)\}$$

$b \in GL_n(\breve{\mathbb{Q}}_p) \rightsquigarrow V_b = (\breve{\mathbb{Q}}_p^n, b\sigma)$ isocrystal / $\breve{\mathbb{Q}}_p$.

$x \in \mathrm{Grass}(n,r)(C) \rightsquigarrow \mathrm{Fil}_x V_{b,C}$ filtration.

$(b, x) \rightsquigarrow (V_b, \mathrm{Fil}_x V_{b,C}) \in \mathrm{Fil}\,\mathrm{Isoc}_{C/\breve{\mathbb{Q}}_p}$.

$\rightsquigarrow HN(b,x) :=$ HN vector of $(V_b, \mathrm{Fil}_x V_{b,C})$

This gives the HN-stratification

$$\mathrm{Grass}(n,r)(C) = \coprod_\lambda \mathrm{Grass}(n,r,b)^{HN=\lambda}.$$

The weakly admissible locus

$$\mathrm{Grass}(n,r,b)^{wa} := \text{semistable objects}$$

$$= \mathrm{Grass}(n,r,d)^{HN=\lambda_0} \quad (\lambda_0 \text{ central})$$

<u>Examples</u> (1) V_b simple isocrystal $\mathrm{Grass}(n,r)$

$$\mathrm{Grass}(n,r,b)^{wa} = \mathrm{Grass}(n,r)$$

(2) $b = \begin{pmatrix} 1 & \\ & p \end{pmatrix}$, $\mathrm{Grass}(2,1) = \mathbb{P}^1$,

$$\mathrm{Grass}(2,1)^{wa} = \mathbb{A}^1, \quad \mathrm{Grass}(2,1)^{HN=(1,-1)} = pt.$$

§3 HN-stratification on B_{dR}-Grassmannian

Grass(n,r) = flag var $\mathcal{F}(GL_n,\mu)$ with $\mu=(1^{(r)},0^{(n-r)})$ minuscule.

Generalization $GL_n \rightsquigarrow$ red gp $G/\mathbb{Q}_p$

minuscule cochar $\mu \rightsquigarrow$ arbitrary cochar μ.

Grass$(n,r) \rightsquigarrow Gr^{B_{dR}}_{G,\mu}$ B_{dR}-Grassmannian.

Filtrations $\rightsquigarrow$ lattices

FilIsoc $\rightsquigarrow$ LattIsoc.

Affine Grass: $Gr_G(\mathbb{C}) = G(\mathbb{C}((t)))/G(\mathbb{C}[[t]])$ ind-complex analytic var.

$\mathbb{C} \rightsquigarrow C$: $\mathbb{C}((t)) \rightsquigarrow B_{dR}(C)$ = frac of $B^+_{dR}(C) \simeq C((\xi))$.

$\mathbb{C}[[t]] \rightsquigarrow B^+_{dR}(C)$ = CDVR w/ res field C, uniformizer ξ.

$B^+_{dR}(C) \simeq \mathbb{C}[[\xi]]$

$Gr^{B_{dR}}_G(C) = G(B_{dR}(C))/G(B^+_{dR}(C))$ has a diamond structure.

$Gr_{G,\mu}(C) \subseteq Gr_G(C)$ μ: bound condition.

Example (1) $G=GL_n$, $Gr_G(C) \xleftrightarrow{1:1}$ lattices in $B_{dR}(C)^n$.

$\cup$ $\qquad$ $\cup$

$Gr_{G,\mu}(C) \xleftrightarrow{1:1}$ lattices in $B_{dR}(C)^n$ s.t. μ holds

(2) $\mu=(1^{(r)},0^{(n-r)})$.

$\Xi \in Gr_{G,\mu}(C) \iff B^+_{dR}(C)^n \overset{(r)}{\subseteq} \Xi \subseteq \xi^{-1}B^+_{dR}(C)^n$

Have Bialynicki-Birula map

$$BB: Gr_{G,\mu}(C) \xrightarrow{\sim} \mathcal{F}(G,\mu)^{(C)} = \mathrm{Grass}(n,r)(C)$$

$$\Xi \longmapsto \Xi/B^+_{dR}(C)^n \subseteq C^n.$$

Caraiani-Scholze: if μ is minuscule,

then BB is an isom.

$\mathrm{LattIsoc}_{C/\breve{\mathbb{Q}}_p} := \{(V, \Xi)\}$

isocrystal $/\breve{\mathbb{Q}}_p$, lattice in $V \otimes B_{dR}(C)$.

$\mathrm{rk}(V, \Xi) := \dim_{\breve{\mathbb{Q}}_p} V$, $\deg(V, \Xi) := \deg \Xi - \dim V$.

where $\Xi \in \mathrm{Gr}_{G,\mu}(C)$, $\deg \Xi := |\mu|$.

We have HN-formalism for any object. Can associate a HN-vector $HN(V, \Xi)$.
The semistable objects are called weakly admissible.

<u>Thm</u> (Chen-Tong) In $\mathrm{LattIsoc}_{C/\breve{\mathbb{Q}}_p}$, the weakly admissible objects are compatible w/ tensor product.

<u>Rem</u> (1) For $\mathrm{FilIsoc}_{C/\breve{\mathbb{Q}}_p}$ the same result is proved by Faltings & Totaro.

(2) If we restrict $\mathrm{LattIsoc}_{C/\breve{\mathbb{Q}}_p}$ to

$$\mathrm{LattIsoc}_{C/\breve{\mathbb{Q}}_p, 0} := \{(V, \Xi) \mid V \text{ is of slope } 0\}.$$

This result is proved by Cornut-Peche

<u>Application</u> We can use Tannakian formalism to define HN-stratification on B_{dR}-Grassmannian $\mathrm{Gr}_{G,\mu} = \coprod_\lambda \mathrm{Gr}_{G,\mu,b}^{HN=\lambda}$.

<u>History</u> (1) When μ is minuscule,

Dat-Orlik-Rapoport $\mathcal{F}(G,\mu)$.

(2) $b=1$ or $G = GL_n$. Nguyen-Viehmann, Shen

(3) (G, b, μ) arbitrary, Chen-Tong.

<u>Prop</u> We know (1) the non-emptyness of $\mathrm{Gr}_{G,\mu,b}^{HN=\lambda}$.

(2) dim formula for $\mathrm{Gr}_{G,\mu,b}^{HN=\lambda}$.

We want to compare the 3 stratifications

(1) HN stratification on $\mathrm{Gr}_{G,\mu}$: stratum $\mathrm{Gr}_{G,\mu,b}^{HN=\lambda}$

(2) Newton stratification on $\mathrm{Gr}_{G,\mu}$: stratum $\mathrm{Gr}_{G,\mu,b}^{New=\lambda}$

(3) HN stratification on $\mathcal{F}(G,\mu)$: stratum $\mathcal{F}(G,\mu,b)^{HN=\lambda}$.

(Dat-Orlik-Rapoport).

$BB : \mathrm{Gr}_{G,\mu} \longrightarrow \mathcal{F}(G,\mu)^{\diamond}$ induces a bijection on classical pts

$\mathcal{F}(G,\mu) = G/P_\mu$ (Fargues-Fontaine, Nguyen-Viehmann.)

P_μ: parabolic subgp.

<u>Prop</u> (a) These 3 stratifications are the same on classical pts.

(b) (1) vs. (2) : $HN(b,x) \leq New(b,x)$.

(1) vs. (3) : $HN(b,x) \leq HN(b, BB(x))$.