Relative Langlands duality via Howe duality

Wee Teck Gan

Two themes: • Howe duality as relative Langlands duality
(HD) (RLD)

• HD from RLD.

<u>RLD</u> Study of periods.

$H \subset G$ spherical subgp. i.e. $X = H\backslash G$ spherical var.

$\chi : H \longrightarrow \mathbb{C}^\times$.

<u>Locally</u> For $\pi \in \mathrm{Irr}\, G$, have $\mathrm{Hom}_H(\pi, \chi)$ with $\dim < \infty$.

Q: Is $\dim \mathrm{Hom}_H(\pi, \chi) \neq 0$?

By Frob reciprocity,

$$\mathrm{Hom}_H(\pi, \chi) \cong \mathrm{Hom}_G(\pi, \mathrm{Ind}_H^G \chi) \quad \mathrm{Ind}_H^G \chi = C^\infty(H, \chi\backslash G)$$

$$\cong \mathrm{Hom}_G(\mathrm{Ind}_H^G \chi, \pi^\vee). \quad \mathrm{Ind}_H^G \chi = C_c^\infty(H, \chi\backslash G).$$

<u>Problem</u> Spectral decomp of $C_c^\infty(H, \chi\backslash G)$.

<u>Globally</u>

$$\begin{array}{ccc} \pi \subseteq \mathcal{A}_{cusp}(G) & \ni & f \\ \downarrow & & \downarrow \\ \mathbb{C} & & \int_{[H]} f \cdot \bar{\chi} \end{array} \rightsquigarrow \text{Q: Is } P_{H,\chi}|_\pi \neq 0\text{?}$$

<u>An expected (vague) answer</u>

(Say π is (H,χ)-distinguished if $\dim \mathrm{Hom}_H(\pi, \chi) \neq 0$.)

(1) (H, χ)-distinguished $\pi \in \mathrm{Irr}\, G$ are functorial lifts from a smaller subgp G_X.

(2) $P_{H,\chi}|_\pi \neq 0 \iff \pi$ is a lift of σ from G_X

and some L-fcn of σ does not vanish at some s_0.

<u>More precise conjectures</u> [SV]

$X = (H\backslash G, \chi) \rightsquigarrow X^\vee \times SL_2 \longrightarrow G^\vee$ with $V_X \circlearrowleft X^\vee$.

$\uparrow$ Langlands dual gp of X

If $\pi \in \mathrm{Irr}\, G$ is of Arthur type,

with A-parameter $\psi : W_F' \times SL_2(\mathbb{C}) \longrightarrow G^\vee(\mathbb{C})$

$X^\vee \times SL_2 \xrightarrow{\iota} G^\vee(\mathbb{C})$

then π is (H,χ)-dist $\Rightarrow \psi$ factors through ι.

If $G_X^\vee = X^\vee$, can get $\psi' : W_F' \longrightarrow X^\vee \longrightarrow GL(V_X)$

$\rightsquigarrow \sigma \in \mathrm{Irr}\, G_X \rightsquigarrow L(s, \sigma, V_X)$.

<u>E.g.</u> · Whittaker: $X = (U, \psi)\backslash G$ with $U \subseteq B$ & $\psi : U \to \mathbb{C}^\times$ generic.

· Shalika period: $G = GL_{2n}$.

$$\left\{ \left(\begin{array}{c|c} A & 0 \\ \hline 0 & A \end{array}\right), \left(\begin{array}{c|c} I & * \\ \hline * & I \end{array}\right) \right\} = H = GL_n^\Delta \ltimes M_{n\times n} \quad (A, x)$$

$$GL_n^\Delta \ltimes M_{n\times n} \to \mathbb{C}, \quad (A,x) \mapsto \psi(\pi(x))$$

$\rightsquigarrow X^\vee = Sp_{2n} \hookrightarrow G^\vee = GL_{2n}$.

· Symplectic period: $Sp_{2n}\backslash GL_{2n} = X$.

$$X^\vee = GL_n \longrightarrow G^\vee = GL_{2n}.$$
$$\times \qquad \times$$
$$SL_2 \qquad SL_2$$

· Gross-Prasad: $X = \Delta SO(2n)\backslash SO(2n) \times SO(2n+1)$, $G = SO(2n+1)$.

$\Rightarrow X^\vee = G^\vee$.

<u>Howe duality</u> $G = O(V) \times Sp(W) \longrightarrow Sp(V \otimes W) \circlearrowright$ Weil rep'n Ω

Problem: Spectral decomp of Ω as G-mod.

<u>HD Thm</u> Ω is strongly multi-free,

i.e. (i) $\forall\ \Pi = \pi \otimes \sigma \in \operatorname{Irr} G$, $\dim \operatorname{Hom}_G(\Omega, \pi) \le 1$.

(ii) If $\operatorname{Hom}_G(\Omega, \pi \otimes \sigma) \neq 0$, $\operatorname{Hom}_G(\Omega, \pi \otimes \sigma') \neq 0$,

then $\sigma \cong \sigma'$.

$\rightsquigarrow$ get a map $\theta : \underset{\pi}{\overset{}{\operatorname{Irr} O(V)}} \longrightarrow \operatorname{Irr} Sp(W) \cup \{0\}$.

<u>Adams Conj</u>: (Moeglin, Bakić–Hansen, Chen–Zou).

If $\Pi \in \operatorname{Irr} G$ of Arthur type, with A-parameter $\psi : W_F' \times SL_2 \to G^\vee$, $G^\vee = O(2m) \times SO(2n+1)$ $(m \le n)$

$\Pi = \pi \otimes \sigma$ occurs in Ω

$\Rightarrow \psi$ factors as

$$
\begin{tikzcd}
W_F' \times SL_2 \arrow[rr, "\psi"] \arrow[dr] & & G^\vee = O(2m) \times SO(2n+1) \\
& X^\vee \times SL_2 \arrow[ur] \arrow[r] & O(2n) \times SO(2n+1-2m) \arrow[u]
\end{tikzcd}
$$

$X^\vee = O(2m)$

Q Is there a large context that encompasses both?

[BZSV] One is looking at spectral decomp of some natural G-mods.

(1st insight) namely, those arising by "quantization" of certain $\underbrace{\text{Hamiltonian } G\text{-var}}_{\text{(symplectic } G\text{-var with moment map)}}$.

E.g. (1) $X = H\backslash G \circlearrowleft G \rightsquigarrow$ cotan bundle T^*X

$C_c^\infty(X)$ or $L^2(X) \xleftarrow{\text{quantization}} T^*X$

$C_c^\infty(X) \circlearrowleft G$, $L^2(X) \circlearrowleft G$

(2) When $G = O(V) \times Sp(W) \circlearrowleft V \otimes W$,

symplectic vs. $\xrightarrow{\text{quantization}}$ Weil rep'n Ω.

Theme. HD as RLD.

Q: Which Hamiltonian G-var to consider?

- Hyperspherical var: $M \curvearrowleft G$ which is s.t.
  - affine & smooth
  - equipped with commuting $\mathbb{G}_m$-action (graded, canonical)
  - multi-free / co-isotropic: $\mathcal{O}(M)^G$ Poisson commutative. (Guillemin–Steinberg).
  - connected generic stabilizer: Poisson alg

Structure thm Hyperspecial M arises from:

- $H \times SL_2 \xrightarrow{\iota} G$ with $H \subseteq Z_G(\iota(SL_2))$ spherical.
- symplectic H-v.s., say S.

For simplicity, assume

- $H = Z_G(\iota(SL_2))$ & $S = 0$.

In this case, data depends only on $\underbrace{SL_2 \to G}$, i.e. a unipotent conj class e in G.

$$\rightsquigarrow \mathfrak{sl}_2 = \langle h, e, f\rangle \to \mathfrak{g}$$

$$\rightsquigarrow M = G \times^H (f + \mathfrak{g}^e/\mathfrak{h})$$
$$= [G \times (f + \mathfrak{g}^e/\mathfrak{h})]/H, \quad \mathfrak{g}^e = \{x \in \mathfrak{g} \mid [x,e] = 0\}.$$

$f + \mathfrak{g}^e$ = Slodowy slice of e.

Quantization of M

$e \rightsquigarrow$ generalized Gelfand-Graev / Whittaker mod

$$\Pi_{M_e} = \mathrm{ind}_{H\cdot \underline{U}^+}^{G} \Psi_f .$$

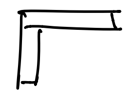
Under $\mathrm{ad}(h)$ on $\mathfrak{g} = \bigoplus_i \mathfrak{g}_i$, $\mathfrak{g}_i = \{x \in \mathfrak{g} \mid [h,x] = iX\}$.

$\rightsquigarrow \mathfrak{h} \subseteq \mathfrak{g}_0$, and $\mathfrak{p} = \bigoplus_{i \geq 0} \mathfrak{g}_i \supseteq \mathfrak{u} = \bigoplus_{i \geq 1} \mathfrak{g}_i \supseteq \mathfrak{u}^+ = \bigoplus_{i \geq 2} \mathfrak{g}_i$.

parabolic

<u>Prop</u> (Work of Bryan Wang)

Young tableux



M_e hyperspherical for $G = Sp$ or SO

$\iff$ e is of hook type or $e \leftrightarrow (2^k)$ (Shalika)

or a small number of cases.

Q: If e is of hook type, what should it be?

<u>2nd Insight</u> On dual side, have

$$(X^\vee \times SL_2 \xrightarrow{2} G^\vee,\ V_X \supseteq X^\vee)$$

$$G^\vee \circlearrowleft \qquad \uparrow W_F' \times SL_2.$$

$$\rightsquigarrow M^\vee = G^\vee \overset{X^\vee}{\times} V_X \quad (\text{if } z(SL_2) = 1)$$

Proposed answer rephrased: W_F-action on $M^\vee$ has a fixed pt.

<u>Conjecture</u> There is a involutive duality on {hyperspherical vars}

$M \mapsto M^\vee$ s.t.

$$\begin{tikzcd} M \arrow[d, "\text{quantize}"'] \arrow[dr] & M^\vee \arrow[d] \arrow[dl] \\ \Pi_M & \Pi_{M^\vee} \end{tikzcd} \quad \text{alg.}$$

Eg. • Whittaker period $M = T^*(\mathcal{U}, \psi \backslash G) \longleftrightarrow M^\vee = pt$.

• $M = T^*(Sp_{2n} \backslash GL_{2n}) \longleftrightarrow (\underset{X^\vee}{\underset{\|}{GL(n)}} \times SL_2 \longrightarrow GL_{2n})$, $\underset{M^\vee}{\wr}$ Shalika period.

• Gross-Prasad $\longleftrightarrow$ Howe duality.