Rankin-Selberg motives of anticyclotomic extensions and Iwasawa main conjecture

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Take $f = q + a_2 q^2 + a_3 q^3 + \cdots \in \mathbb{Z}[[q]]$,

cuspidal newform of wt 2 & s.t. $p \nmid a_p$ for some p.

$\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{Gal}(\mathbb{Q}(\mu_{p^\infty})/\mathbb{Q}) \simeq \mathbb{Z}_p^\times \xrightarrow[\text{finite}]{\chi} \mathbb{C}^\times$.

Consider $L(f \otimes \chi, s)$ centered at 1.

<u>Q</u> Can this glue to some p-adic fcn (w/ sp value at $s=1$)

fcn on the space of all p-adic chars of $\mathbb{Z}_p^\times$.

Have $\mathscr{L}_p^+(f)$ p-adic measure on $\mathbb{Z}_p^\times \iff$ element in $\mathbb{Z}_p[[\mathbb{Z}_p^\times]] \otimes_{\mathbb{Z}_p} \mathbb{Q}_p =: \mathbb{Q}_p[[\mathbb{Z}_p^\times]]^\circ$.

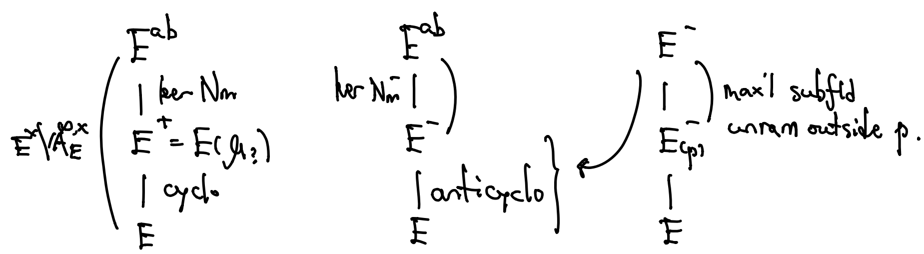
s.t. for χ finite, $\mathscr{L}_p^+(f)(\chi) \sim L(1, f \otimes \chi)$.

Let $E/\mathbb{Q}$ imag quad, E^{ab} max abelian ext'n of E

$\rightsquigarrow \mathrm{Gal}(E^{ab}/E) \simeq E^\times \backslash \mathbb{A}_E^{\infty,\times}$ by CFT.

Have $E^1 \backslash \mathbb{A}_E^{\infty,1} \to E^\times \backslash \mathbb{A}_E^{\infty,\times} \xrightarrow{\mathrm{Nm}} \mathbb{Q}^\times \backslash \mathbb{A}_\mathbb{Q}^{\infty,\times}$.

$\mathrm{Nm}^- : x \mapsto x/\bar{x}$.



Denote by $\Gamma_{E,p}^- := \mathrm{Gal}(E^-_{(p)}/E)$.

<u>Fact</u> $\exists\ \mathscr{L}_p^-(f) \in \mathbb{Q}_p[[\Gamma_{E,p}^-]]^\circ$ s.t. $\forall$ finite $\chi : \Gamma_{E,p}^- \to \mathbb{C}^\times$,

$\mathscr{L}_p^-(f)(\chi) \sim L(1, f, \chi) \leftarrow$ Rankin-Selberg L-fcn.

E/F CM ext'n of number fields.

$\mathfrak{p}$ p-adic place of F splitting in E.

$\mathbb{L}/\mathbb{Q}_p$ finite ext'n.

<u>Def'n</u> A <u>relevant rep'n</u> π of $GL_N(\mathbb{A}_E^\infty)$ w.c. in $\mathbb{L}$ is an adm rep'n of $GL_N(\mathbb{A}_E^\infty)$

s.t. $\forall\, \iota: \mathbb{L} \to \mathbb{C}$, $\pi_\infty^{\min} \otimes {}_\iota\pi$ is

an isobaric sum of $d(\pi)$ mutually nonequiv conj self-dual cuspidal rep'ns.

(isobaric sum: sth. like Eisenstein sum.)

Take $n \geq 1$ and consider π_n, π_{n+1} relevant rep'ns of

$$GL_n(\mathbb{A}_E^\infty) \ \& \ GL_{n+1}(\mathbb{A}_E^\infty) \quad \text{respectively.}$$

$\rightsquigarrow$ Write $\pi = \pi_n \boxtimes \pi_{n+1}$ $\rightsquigarrow$ $\epsilon(\pi) \in \{\pm 1\}$ root number of π.

<u>Thm</u> Suppose that $\pi_{\mathfrak{p}} = \pi_{n,\mathfrak{p}} \boxtimes \pi_{n+1,\mathfrak{p}}$ is semistable & ordinary.

$\exists!\ \mathscr{L}_{\mathfrak{p}}^\circ(\pi) \in \mathbb{L}[\![\tilde{\Gamma}_{E,\mathfrak{p}}]\!]^\circ$ s.t. $\forall$ fin char $\chi: \tilde{\Gamma}_{E,\mathfrak{p}} \to \bar{\mathbb{L}}^\times$ of conductor $\mathfrak{f} > 0$ at $\mathfrak{p}$.

and $\forall\, \iota: \bar{\mathbb{L}} \to \mathbb{C}$,

$$\iota\, \mathscr{L}_{\mathfrak{p}}^\circ(\pi)(\chi) = \left(\frac{q^{\frac{n(n+1)(2n+1)}{6}}}{\iota \cdot \lambda(\pi_{\mathfrak{p}})} \right)^{\mathfrak{f}} \cdot \frac{\Delta_{n+1} \cdot L(\frac{1}{2}, \iota(\pi_n \otimes \chi) \times \pi_{n+1})}{2^{d(\pi_n)+d(\pi_{n+1})} L(1, \iota\pi_n, As^{(-1)^n}) \cdot L(1, \iota\pi_{n+1}, As^{(-1)^{n+1}})}$$

q = residue char of $\mathfrak{p}$, $\lambda(\pi_{\mathfrak{p}}) \in \mathcal{O}_{\mathbb{L}}^\times$.

When $\epsilon(\pi) = 1$, by GGP. $\exists!$ tot pos-def herm space V/E of dim n

$$G = U(V) \times U(V \oplus \mathbb{1}) \xleftarrow{\Delta} H := U(V) \qquad /F$$

& an adm rep'n π of $G(\mathbb{A}_F)$ w.c. in $\mathbb{L}$

s.t. $\mathrm{Hom}_{H(\mathbb{A}_F)}(\pi, \mathbb{L}) \neq 0$.

Denote $V_\pi := \mathrm{Hom}_{G(\mathbb{A}_F)}(\pi, \mathcal{C}(G(F) \backslash G(\mathbb{A}_F^\infty)))$.

Conj (N0) $\mathcal{L}_p^0(\pi) \neq 0 \iff V_\pi \neq 0$.

Rmk $V_\pi \neq 0 \iff$ all "isobaric" ϵ-factor is 1.

$\pi \rightsquigarrow \mathbb{W}_\pi$ w.c. in $\mathbb{L}$ of $\mathrm{Gal}(\bar{E}/E)$ s.t. $\mathbb{W}_\pi^\vee \simeq \mathbb{W}^c(-1)$.

$$E_{\mathfrak{p}}^- \supseteq E_{\mathfrak{p}}^{(f)} \supseteq \cdots \supseteq E_{\mathfrak{p}}, \quad f > 0.$$

$$\varinjlim_f H^1_{fin}(E_{\mathfrak{p}}^{(f)}, \mathbb{W}_\pi) =: H^1_{fin}(E, \mathbb{W}_\pi)_{\mathfrak{p}}$$

$$\cup\!\!\!|$$

$$H^1_{fin}(E, \mathbb{W}_\pi)^\circ_{\mathfrak{p}} = \bigcup_{\text{lattice of } \mathbb{W}_\pi} H^1_{fin}(E, \mathbb{W}_\pi^\circ)_{\mathfrak{p}}$$

It turns out: $H^1_{fin}(E, \mathbb{W}_\pi)^\circ$ compact $\mathbb{L}[\Gamma_{E,\mathfrak{p}}^-]^\circ$-mod.

Conj (I0) $\mathrm{char}(H^1_{fin}(E, \mathbb{W}_\pi)^\circ)$ is generated by $\mathcal{L}_p^0(\pi^\vee)$.

Thm (Liu-Tian-Xiao) Under some conditions.

$$\mathcal{L}_{\mathfrak{p}}^0(\pi^\vee) \in \mathrm{char}(H^1_{fin}(E, \mathbb{W}_\pi)^\circ).$$

$\epsilon(\pi) = -1$, $E \overset{\varphi}{\hookrightarrow} \mathbb{C}$, $\exists!\, V$ of sgn $(n-1, 1)$ at φ & $(n, 0)$ away from φ.

$$G \rightsquigarrow \{X_K\}_{K \subseteq G(\mathbb{A}_F^\infty)}, \quad H \rightsquigarrow \{Y_{K_H}\}_{K_H \subseteq H(\mathbb{A}_F^\infty)}$$

$$V_\pi := \mathrm{Hom}_{G(\mathbb{A}_F^\infty)}(\pi^\vee, H^{2n-1}(\bar{X}_K, \mathbb{L}(n))) \circlearrowleft \mathrm{Gal}(\bar{E}/E).$$

$$\pi \longrightarrow \mathrm{Hom}_{\mathrm{Gal}}(H^{2n-1}(\bar{X}_K, \mathbb{L}(n)), V_\pi)$$

$\forall \varphi \in \pi$, s.t. φ is ordinary vec at $\mathfrak{p}$, $Z_\varphi \in H^1_{fin}(E, \mathbb{W}_\pi)^\circ$.

$$(\cdot,\cdot) : H^1_{fin}(E, \mathbb{W}_\pi)^\circ \times H^1_{fin}(E, \mathbb{W}_{\pi^\vee})^\circ \longrightarrow \mathbb{L}[\Gamma_{E,\mathfrak{p}}^-]^\circ \otimes_{\mathbb{Z}_p} \Gamma_{F,\mathfrak{p}}$$

Define $\mathcal{L}_{\mathfrak{p}}^1(\pi) := c(\varphi, \varphi^\vee)(Z_\varphi, Z_{\varphi^\vee})$ for $\varphi \in \pi$, $\varphi^\vee \in \pi^\vee$

s.t. $\mathcal{L}_{\mathfrak{p}}^1(\pi)$ indep of choices of φ & $\varphi^\vee$.

Conj (N1) $\mathcal{L}_{\mathfrak{p}}^1(\pi) \neq 0 \iff V_\pi \neq 0$.

Conj (I1) $\mathrm{char}(H^1_{fin}(E, \mathbb{W}_\pi)^\circ / Z_\pi)$ is generated by $\{\ell\, \mathcal{L}_{\mathfrak{p}}^1(\pi) : \ell : \Gamma_{F,\mathfrak{p}} \to \mathbb{Z}_p\}$.

Z_π submod generated by Z_φ for all φ.

Expectation $\mathcal{L}_p^E(\pi) \sim L(\frac{1}{2}, \pi_n \boxtimes \pi_{n+1} \otimes \chi)$, where $\chi : \Gamma_{E,p} \to \bar{\mathbb{L}}^\times$.

Conj (Interpolation) $\mathcal{L}_p^\dagger(\pi) = d\mathcal{L}_p^E(\pi)|_{\Gamma^-_{E,p}}$.

(When $n=1$, this is the classical Rankin-Selberg.)