Harris–Venkatesh plus Stark

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§1 Stark and Harris–Venkatesh

Setup f modular form, wt 1, level N. E fin ext'n of $\mathbb{Q}$.

(By Deligne–Serre) $\rightsquigarrow \rho_f : \mathrm{Gal}(E/\mathbb{Q}) \longrightarrow GL_2(\mathbb{C})$.

$\rightsquigarrow \mathrm{Ad}(\rho_f)$ 3-dim'l rep'n, factors through $PGL_2(\mathbb{C}) = SO_3(\mathbb{C})$.

fin image: cyclic gp D_{2n}, A_4, S_4, A_5

Eisenstein dihedral / exotic.

$$\begin{tikzcd}[column sep=small]
\text{(A)} & & ② \; L'(\mathrm{Ad}(\rho_f), 0) \text{ centered at } 1/2 \arrow[dl, leftrightarrow] \arrow[dr, leftrightarrow] & \\
& ③ \; \|f\|^2_{\mathbb{R}} \arrow[rr, leftrightarrow, "\text{Stark} / \mathbb{C}"] & & \text{units in } \mathcal{O}_E^{\times} \; ① \text{ (regulator)} \\
\text{(B)} & \|f\|^2_{\mathbb{F}_p^{\times}} \; ⑤ \arrow[rr, leftrightarrow, "\text{Harris-Venkatesh} / \mathbb{F}_p^{\times}"] & & \text{reduction of units in } \mathcal{O}_E^{\times} \text{ mod } p \; ④
\end{tikzcd}$$

① Dual units $\mathcal{U}(\mathrm{Ad}(\rho_f)) := \mathrm{Hom}_{\mathrm{Gal}(E/\mathbb{Q})}(\mathrm{Ad}(\rho_f), \mathcal{O}_E^{\times})$.

Regulator w archimedean place

$\rightsquigarrow x_w := 2\rho_f(\mathrm{Frob}_w) - \mathrm{Tr}(\rho_f(\mathrm{Frob}_w)) \in \mathrm{Ad}(\rho_f)$.

$\rightsquigarrow \mathrm{Reg}_{\mathbb{R}} : \mathcal{U}(\mathrm{Ad}(\rho_f)) \longrightarrow \mathcal{O}_E^{\times} \; (\otimes \mathbb{Z}[\chi_{\mathrm{Ad}(\rho_f)}])$

$\log \circ$ (evaluation at x_w)

② Conjecture (Stark, 1971, 1975, 1976, 1980)

There is a unique $u_{\mathrm{stark}} \in \mathcal{U}(\mathrm{Ad}(\rho_f))$ s.t.

$$\begin{aligned} L'(\mathrm{Ad}(\rho_f), 0) &= \mathrm{Reg}_{\mathbb{R}}(u_{\mathrm{stark}}) \\ &= \sum_{\theta \in \mathrm{Gal}(E/\mathbb{Q})} \chi_{\mathrm{Ad}(\rho_f)}(\theta) \cdot \log(\underset{\underset{(\exists! \, \varepsilon \in \mathcal{O}_E^{\times})}{\uparrow}}{\varepsilon^{\theta}}) . \end{aligned}$$

③ Petersson norm $\|f\|^2_{\mathbb{R}} := \int_{X_0(N)} |f|^2 \cdot \frac{dx\,dy}{y^2} = c \cdot L'(\mathrm{Ad}(\rho_f), 0)$.

④ <u>Regulator in $\mathbb{F}_p^\times$</u>

Fix a place w of E above p.

$\rightsquigarrow x_w := 2\rho_f(\mathrm{Frob}_w) - \mathrm{Tr}(\rho_f(\mathrm{Frob}_w)) \in \mathcal{U}(\mathrm{Ad}(\rho_f))$.

Evaluation at x_w : $\mathcal{U}(\mathrm{Ad}(\rho_f)) \longrightarrow (\mathcal{O}_E^\times)^{\mathrm{Frob}_w}$,

$$\hookrightarrow\!\!\!\!\!\rightarrow \mathrm{im} \subseteq \mathbb{Z}_p^\times .$$

$\rightsquigarrow \mathrm{Reg}_{\mathbb{F}_p^\times} : \mathcal{U}(\mathrm{Ad}(\rho_f)) \longrightarrow \mathbb{F}_p^\times$ reduction modulo ideal of w.

⑤ <u>Shimura class</u>

Let $p \nmid 6N$. There is an étale covering $\begin{array}{c} X_1(p) \\ \downarrow \mathbb{F}_p^\times \\ X_0(p) \end{array}$

$\rightsquigarrow \mathcal{A} \in H^1_{\acute{e}t}(X_0(p), \mathbb{F}_p^\times) = H^1_{\acute{e}t}(X_0(p), \mathbb{Z}/(p-1)\mathbb{Z}) \otimes \mathbb{F}_p^\times$.

Take base change $X_0(p) \otimes \mathbb{Z}/(p-1)\mathbb{Z}$

$\rightsquigarrow$ push-forward $\mathbb{Z}/(p-1)\mathbb{Z} \longrightarrow \mathbb{G}_a$.

$$\rightsquigarrow \mathcal{A}_p \in \underset{\substack{\| \\ \mathrm{Zar}}}{H^1_{\acute{e}t}}(X_0(p) \otimes \mathbb{Z}/(p-1)\mathbb{Z}, \underset{\substack{\| \\ \mathcal{O}}}{\mathbb{G}_a}) \otimes \mathbb{F}_p^\times$$

$$\Updownarrow$$

By Serre duality $\mathcal{A}_p \in \mathrm{Hom}(H^0(X_0(p), \Omega^1), \mathbb{F}_p^\times)$

with $\mathcal{A}_p$: wt 2 mod forms $\longrightarrow \mathbb{F}_p^\times$.

<u>Harris-Venkatesh norm</u> $\|f\|^2_{\mathbb{F}_p^\times} := \mathcal{A}_p(\mathrm{Tr}_p^{Np}(f(z) \cdot f^*(pz)))$

Tr_p^{Np} : level $\Gamma_1(N) \cap \Gamma_0(p)$

<u>Conj</u> (Harris-Venkatesh, 2019)

There is a $u \in \mathcal{U}(\mathrm{Ad}(\rho_f))$ & $m \in \mathbb{N}$ s.t. $p \nmid 6N$, $m \cdot \|f\|^2_{\mathbb{F}_p^\times} = \mathrm{Reg}_{\mathbb{F}_p^\times}(u)$.

<u>§2 Unified conjecture</u>

<u>Conj</u> (H-V plus Stark) There is a unique $u_f \in \mathcal{U}(\mathrm{Ad}(\rho_f))$ s.t.

(1) $\|f\|^2_{\mathbb{R}} = \mathrm{Reg}_{\mathbb{R}}(u_f)$.

(2) $\|f\|^2_{\mathbb{F}_p^\times} = \mathrm{Reg}_{\mathbb{F}_p^\times}(u_f)$, for $p \gg 0$. $\quad (m \in \prod_{p \le p_0} (p-1))$.

Rem (1) u_f is now unique for the HV conj.

(2) require compatibility b/w Stark & H-V.

Thm (Zhang) The H-V plus Stark is true for all imaginary dihedral forms.

Def'n If f is dihedral, then $\rho_f = \mathrm{Ind}_{G_K}^{G_{\mathbb{Q}}}(\chi)$

where χ nontriv fin characters of $\mathrm{Gal}(\bar{K}/K)$,

$K/\mathbb{Q}$ quadratic number field.

$\rightsquigarrow$ Case I: K imaginary $\left(\mathrm{Ad}(\rho_f) = \eta \oplus \mathrm{Ind}_{G_K}^{G_{\mathbb{Q}}}(\xi),\ \xi = \chi/\bar{\chi}.\right)$

Case II: K real.

Rem (1) Stark Conj already known in this case (Stark).

(2) Darmon - Harris - Rotger - Venkatesh (2022):

HV is true for imaginary dihedral f

if χ unramified & D_K odd prime.

(3) Lecouturier.

§3 Overview for the imaginary dihedral forms

	Stark	Harris-Venkatesh	Gross
	$\mathbb{R}$	$\mathbb{F}_p^\times$	$\mathbb{Q}_p$
	Any irred ρ	Adjoint rep'n $\mathrm{Ad}(\rho_f)$	Totally odd rep'n ρ
(imag dihedral)	$\rho = \chi$	DHRV 2022	Darmon-Dasgupta-Pollack 2011
	(Stark 1980)	Zhang 2023	Dasgupta-Kakde-Ventullo 2019

· <u>DHRV</u> $K/\mathbb{Q}$ imag quad, D_K odd prime.

χ nontriv fin char of G_K, χ unramified.

HV conj is triv if p splits in K ($\Rightarrow$ assume p inert in K.)

<u>Chain of equalities</u>

$$\|f\|^2_{\mathbb{F}_p^\times} = \mathcal{A}_p(\mathrm{Tr}^{N_p}_p(f(z)\cdot f^*(pz))) = \langle \mathrm{Tr}^{N_p}_p(f(z)\cdot f^*(pz)), \mathcal{A}_p\rangle = \boxed{\text{Const}}\cdot \mathcal{A}_p(f^{opt}(z,pz)).$$

$$= \langle \Theta_p([1]\otimes[\xi]), \mathcal{A}_p\rangle.$$

(<u>Ingredients</u> $[\cdot]$: Hecke characters $\longrightarrow$ Autom forms

$K^\times\backslash\mathbb{A}_K^\times$ $\qquad$ $B^\times\backslash B_{\mathbb{A}}^\times$.

Θ_p : modular forms on $(B^\times)^{\otimes 2} \longrightarrow S_2(p)$.)

$$\mathrm{RHS} = \langle [1]\otimes[\xi], \Theta_p^*(\mathcal{A}_p)\rangle = \langle [1]\otimes[\xi], \varepsilon_p\rangle$$

$$= \boxed{\text{Const}}\cdot \mathrm{Reg}_{\mathbb{F}_p^\times}(u_\xi).$$

<u>Optimal form</u> $f^{opt}(z_1,z_2)$ two-variable modular form on $X(N)\times X(N)$.

$\in \pi_f$ (cuspidal autom rep'n of GL_2, gen'd by f).

Uniquely determined by $f^{opt}(z,pz) = \Theta_p\langle[1]\otimes[\xi]\rangle$, $\forall p\nmid 6N$.

<u>Multiplicity-one argument in char > 0</u>

$$\frac{\mathcal{P}_{RS}(f\otimes f^*)}{\mathcal{P}_{RS}(f^{opt})} = \frac{\mathcal{P}_{HV}(f\otimes f^*)}{\mathcal{P}_{HV}(f^{opt})} \longleftarrow = \text{“}\|f\|^2_{\mathbb{F}_p^\times}\text{”}$$