# Slopes of modular forms and ghost conjecture

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Let $p \geq 5$ be a prime. $E/\mathbb{Q}_p$ fin ext'n.

$E \supseteq \mathcal{O} \twoheadrightarrow \mathcal{O}/(\varpi) \cong \mathbb{F}$ coeffs.

<u>Classification (by Serre) of 2-dim'l mod p rep'n of $\mathrm{Gal}_{\mathbb{Q}_p}$</u>

Notation: $\mathrm{unr}(\bar{\alpha}) :=$ unram rep of $\mathrm{Gal}_{\mathbb{Q}_p}$ sending geom Frob $\mapsto \bar{\alpha} \in \mathbb{F}^\times$.

$\omega_1 : \mathrm{Gal}_{\mathbb{Q}_p} \twoheadrightarrow \mathrm{Gal}(\mathbb{Q}_p(\mu_p)/\mathbb{Q}_p) \simeq \mathbb{F}_p^\times$ 1st fundamental char.

$\omega_2 : \mathrm{Gal}_{\mathbb{Q}_{p^2}} \twoheadrightarrow \mathrm{Gal}(\mathbb{Q}_{p^2}(\sqrt[p^2-1]{p})/\mathbb{Q}_{p^2}) \simeq \mathbb{F}_{p^2}^\times$ 2nd fundamental char.

<u>Reducible type</u>: $\bar{\rho} = \begin{pmatrix} \mathrm{unr}(\bar{\alpha})\,\omega_1^{a+1} & * \\ 0 & \mathbb{1} \end{pmatrix} \otimes \mathrm{unr}(\bar{\beta})\cdot\omega_1^{b}$, $a \in \{0, \dots, p-2\}$.
$\bar{\alpha}, \bar{\beta} \in \mathbb{F}^\times$.

Call $\bar{\rho}$ generic if $1 \leq a \leq p-4$.

In this case, $\begin{cases} * = 0 & \text{split.} \\ * \neq 0 & \text{nonsplit. unique such ext'n as rep'ns, up to isom.} \end{cases}$

<u>Irred type</u>: $\bar{\rho} = \mathrm{unr}(\bar{\alpha})\cdot\mathrm{Ind}_{\mathrm{Gal}_{\mathbb{Q}_{p^2}}}^{\mathrm{Gal}_{\mathbb{Q}_p}} \omega_2^{p(a+1)}$.

Want $\bar{r} : \mathrm{Gal}_{\mathbb{Q}} \to \mathrm{GL}_2(\mathbb{F})$ irred $\rightsquigarrow$ restr to local datum.

Take a $\bar{\rho} \rightsquigarrow R_{\bar{\rho}} =$ univ deform ring (with a fixed det).

$\rightsquigarrow$ $\begin{array}{c} \mathcal{V} = \text{univ deform} \\ | \\ R_{\bar{\rho}} \end{array}$.

Interested in those $x \in \mathrm{Spec}\, R_{\bar{\rho}}[\frac{1}{p}]$ that are "trianguline".

i.e. $0 \to \mathcal{R}(\delta_+) \to D_{\mathrm{rig}}(\mathcal{V}_x) \to \mathcal{R}(\delta_-) \to 0$.

short exact seq of $(\varphi, \Gamma)$-mod / Robba ring.

where $\delta_\pm : \mathbb{Q}_p^\times \to E^\times$ conti chars.

$\mathcal{R}(\delta_\pm) = \mathcal{R}e_\pm$, $\varphi(e_\pm) = \delta_\pm(p)e_\pm$, $\gamma(e_\pm) = \delta_\pm(\chi_{cycl}(\gamma))e_\pm$.

E.g. Suppose $f \in S_k(\Gamma_0(N))$, $p \nmid N$ normalized eigenform.

$$\rho_f : \mathrm{Gal}_{\mathbb{Q}} \longrightarrow GL_2(E).$$

Suppose $\bar{\rho}_{f,p} \simeq \bar{\rho}$. Then $\rho_f$ is crystalline at $p$.

$$D_{cris}(\rho_f) \circlearrowleft \varphi \text{ Frob}$$

$$\mathrm{char}(\varphi) = x^2 - a_p(f)x + p^{k-1}.$$

If $\alpha, \beta$ = roots of $\mathrm{char}(\varphi)$ $(\alpha \neq p)$,

$\Rightarrow$ $0 \to \mathcal{R}(\delta_+) \to D_{rig}(\rho_{f,p}) \to \mathcal{R}(\delta_-) \to 0$ trianguline

where $\delta_+(p) = \alpha$ or $\beta$, $\delta_-(p) = \beta \cdot p^{1-k}$ or $\alpha \cdot p^{1-k}$.

$\delta_+|_{\mathbb{Z}_p^\times} = \mathrm{triv}$, $\delta_-(a) = a^{1-k}$.

$\exists$ a "moduli" space of trianguline rep'ns.

$$\begin{tikzcd} \mathcal{X}_{\bar{\rho}} := (\mathrm{Spf}\, R_{\bar{\rho}})^{rig} & \mathcal{X}_{\bar{\rho}}^{tri} \arrow[l] \arrow[r, "{(\delta_-,\delta_+)}"] \arrow[rr, bend left, "\delta_+(p)/\delta_-(p)"] \arrow[d, bend right, "\text{wt map}"'] & \mathrm{Hom}(\mathbb{Q}_p^\times, \mathbb{C}_p^\times)^2 & \mathbb{G}_m^{rig}. \\ & \mathrm{Hom}(\mathbb{Z}_p^\times, \mathbb{C}_p^\times). & & \end{tikzcd}$$

$\uparrow$ trianguline : $0 \to \mathcal{R}(\delta_+) \to D_{rig}(V) \to \mathcal{R}(\delta_-) \to 0$.

<u>Main goal</u> Study the geometry of $\mathcal{X}^{tri}$ & the maps wt, $a_p$ map.

Consider the wt space $\mathcal{W} =$ "$\mathrm{Hom}_{cont}(\mathbb{Z}_p^\times, \mathbb{C}_p^\times)$". $\mathbb{Z}_p^\times = \underset{\Delta}{\underset{\|}{(\mathbb{Z}/p\mathbb{Z})^\times}} \times (1+p\mathbb{Z}_p)$.

$$= \hat{\Delta} \times \underbrace{\mathrm{Hom}(1+p\mathbb{Z}_p, \mathbb{C}_p^\times)}_{\text{pro-cyclic gp}} \ni \; 1+p \mapsto 1+w \text{ for } v_p(w) > 0.$$

<u>Theorem</u> (LTXZ) Assume $\bar{\rho}$ reducible, $2 \le a \le p-5$, $p \ge 11$.

For each weight disc $\tilde{\eta} \in \hat{\Delta}$, $\exists$ an explicit combinatorially def'd power series $G_{\bar{\rho},\tilde{\eta}}(w,t) = \sum_n g_n(w) \cdot t^n \in \mathbb{Z}_p[w][[t]]$.

(ghost series of Bergdall-Pollack.)

s.t. $\forall\, w_\star \in M_{a_p}$, $z \in wt^{-1}(w_\star)$.

$$\begin{tikzcd}
X^{tri}_{\bar\eta} \ni z \arrow[rr, "a_p"] \arrow[d, "wt"'] & & \mathbb{G}_m^{rig} \\
\mathcal{W}_{\bar\eta} \quad w_\star = (1+p)^k - 1 . & &
\end{tikzcd}$$

$v_p(a_p(z))$ is a slope of $NP(G_{\bar p,\bar\eta}(w_\star, -))$

Convex hull of $(n, v_p(g_n(w_\star)))$.

Cor For fixed $w_\star$, such $v_p(a_p(z))$ form a discrete set.

Implication $f \in S_k(\Gamma_0(N))$ eigenform $p \nmid N$.

Deligne: $\quad |\alpha|_\infty = 2 \cdot p^{\frac{k-1}{2}}$ $\quad$ ($\alpha, \beta$ roots of $X^2 - a_p(f) X + p^{1-k} = 0$.)

$\ell \nmid pN$, $v_\ell(\alpha) = 0$.

$\rightsquigarrow$ Q: $v_p(\alpha) = ?$

Suppose we are in the same setup for $G$ $(GL_2, U(2)^d, U(1,1)^d)$.

$G \times_{\mathbb{Q}} \mathbb{Q}_p \trianglerighteq H = GL_{2,\mathbb{Q}_p}$.

Localize at $\bar r : \mathrm{Gal}_{\mathbb{Q}} \longrightarrow GL_2(\mathbb{F})$ irred.

s.t. $\bar r|_{\mathrm{Gal}_{\mathbb{Q}_p}}$ is reducible, very generic.

$\rightsquigarrow$ "overconvergent autom forms" and "eigencurve".

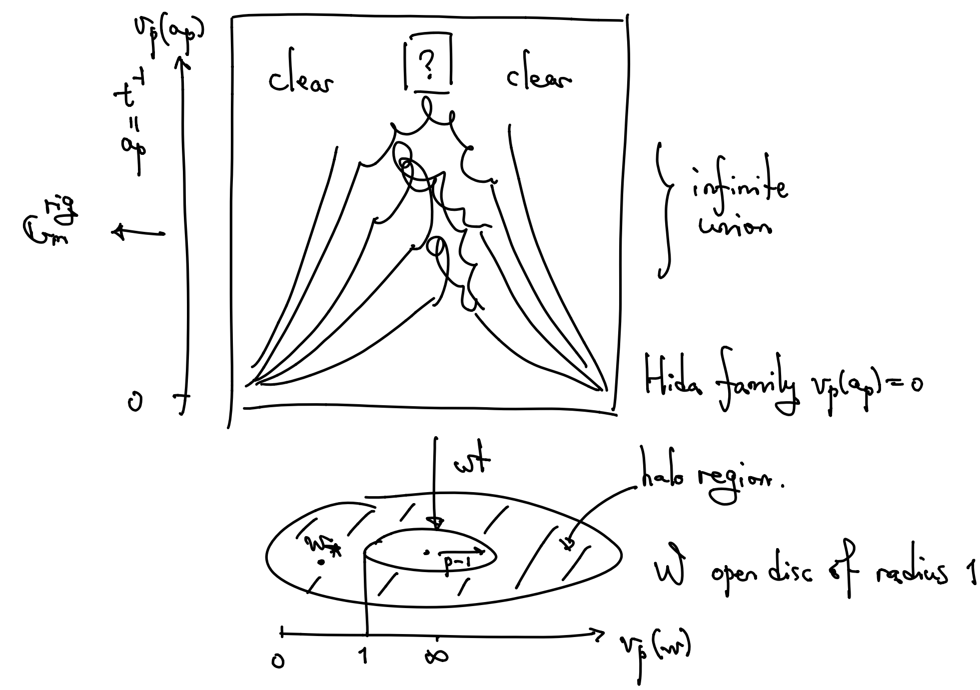
If we assume a technical mod-p-mult one condition,

$$\begin{tikzcd}
Z(x = x_0) \arrow[d, phantom, "\cap|"] & \mathcal{E} \arrow[l, hook'] \arrow[d, phantom, "\cong"] & \\
X_{\bar p} \arrow[u, bend right, "x \in R_{\bar p}"'] & X^{tri} \arrow[l, hook'] \arrow[r, "a_p"] \arrow[d, "nice"] & \mathbb{G}_m^{rig} \\
& \mathcal{W} \times disc_x &
\end{tikzcd}$$

depends on autom data $x_0 \in \Phi$

Geometry of $\mathcal{E}$ (or $\mathcal{X}^{\mathrm{tri}}$)

Picture :




Thm (LTXZ) Given any autom setup as above.

① $\mathcal{W}_{\bar\eta}^{(0,1)} := \{ w_\star \in \mathfrak{m}_{\mathbb{C}_p},\ v_p(w_\star) \in (0,1) \}$

$\mathcal{E}_{\bar\eta}^{(0,1)} := \mathrm{wt}^{-1}(\mathcal{W}_{\bar\eta}^{(0,1)})$ is an infinite disjoint union $= \coprod_{n\geq 0} \mathcal{E}_{\bar\eta,n}^{(0,1)}$.

s.t. (1) $\mathrm{wt}: \mathcal{E}_{\bar\eta,n}^{(0,1)} \xrightarrow{\ \simeq\ } \mathcal{W}_{\bar\eta}^{(0,1)}$

(2) $\forall z \in \mathcal{E}_{\bar\eta,n}^{(0,1)}$, $\dfrac{v_p(a_p(z))}{v_p(\mathrm{wt}(z))} = \deg g_n(w) - \deg g_{n-1}(w)$.

$\left( G_{\bar\rho,\bar\eta}(z) = \sum g_n(w)\cdot t^n \in \mathbb{Z}_p[\![w]\!][\![t]\!]. \right)$

Proof Each $g_n(w) = \prod_{\text{some } k} (w - \underset{\substack{\parallel \\ (1+p)^k - 1}}{w_k})^{m_n(k)}$

note If $v_p(w_\star) \in (0,1)$, then $v_p(g_n(w_\star)) = v_p(w_\star)\cdot \deg g_n$.

$\Downarrow$

$(n, v_p(g_n(w_\star))) = (n, \deg g_n \cdot v_p(w_\star))$.

② $\mathcal{E}_{\bar\eta}^{\text{non-Hida}}$ is irred. (a question of Coleman-Mazur).