A prismatic-étale comparison theorem in the semi-stable case

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§1 Classical cohom comparisons revisit

$K/\mathbb{Q}_p$ fin ext'n, $\mathcal{O}_K$, $k = \mathcal{O}_K/(\pi)$. $W = W(k)$, $K_0 = W[\frac{1}{p}]$.

$X/\mathcal{O}_K$ proper semistable sch. i.e. étale locally of the form

$$\operatorname{Spec} \mathcal{O}_K[T_0, \cdots, T_r, \cdots, T_d]/(T_0 \cdots T_r - \pi).$$

- p-adic étale coh: $H^i_{ét}(X_{\bar{K}}, \mathbb{Q}_p) \circlearrowleft \mathrm{Gal}_K$
- log-crys coh (Hyodo-Kato, Grothendieck, Berthelot):

$$H^i_{cris}(X_k/K_0) \quad K_0\text{-v.s.} \circlearrowleft \varphi \text{ Frob. } N \text{ nilpotent}, \; N\varphi = p\varphi_N.$$

<u>Thm</u> There exists a canonical isom

$$H^i_{ét}(X_{\bar{K}}, \mathbb{Q}_p) \otimes_{\mathbb{Q}_p} B_{st} \xrightarrow{\sim} H^i_{HK}(X_k/K_0) \otimes_{K_0} B_{st}$$

compatible w/ Gal_K-action, φ, N.

where B_{st}: huge period ring def'd by Fontaine
$\circlearrowleft$
$\mathrm{Gal}_K, \varphi, N$.

<u>Rem</u> This was a conj by Fontaine-Janssen.

proved by: Tsuji (1999) & Faltings, Niziol, Beilinson, Bhatt, ...

<u>Q</u> · $\mathbb{Z}_p$-coefficients?

· More general $\mathbb{Z}_p$-local systems?

E.g. $\mathcal{A}$ ab sch of dim $g \geq 1$.

$$\begin{array}{c} \mathcal{A} \\ \downarrow f \\ X \end{array}$$

- $\mathbb{L} := R^1 f_{K,ét,*}(\mathbb{Z}_p)$, $f_{K,ét}: \mathcal{A}_{K,ét} \to X_{K,ét}$.
- $\mathcal{E} := R^1 f_{cris,*}(\mathcal{O}_{\mathcal{A}_k/W})$, $f_{cris}: (\mathcal{A}_k/W)_{log\text{-}cris} \to (X_k/W)_{log\text{-}cris}$.

<u>Q</u> Does there exist a comparison thm for

$$H^i_{ét}(X_{\bar{K}}, \mathbb{L}) \longleftrightarrow H^i_{log\text{-}cris}((X_k/W)_{crys}, \mathcal{E})\ ?$$

<u>The crys case</u> : · Faltings (1990)

· Andreatta - Iovita (2013, $K = K_0$).

· Tan - Tong (2019, $K = K_0$).

· Bhatt - Morrow - Scholze (2018) & Bhatt - Scholze (2021).

· Guo - Reinecke (2022) :

p-adic comparison for crystalline $\mathbb{Z}_p$-loc system.

$X / \mathrm{Spf}\, \mathcal{O}_K$ proper smooth.

$$\begin{array}{cc} \mathbb{L} & \mathrm{Loc}^{\mathrm{crys}}_{\mathbb{Z}_p}(X_{K,\acute{e}t}) := \{\text{crystalline } \mathbb{Z}_p\text{-loc on } X_{K,\acute{e}t}\} \\ \downarrow & \downarrow \\ (\mathcal{E}_0, \varphi_0, \mathrm{Fil}^\cdot \mathcal{E}_K) & \mathrm{IsoFil}(X) := \left\{ (\mathcal{E}_0, \varphi_0, \underline{\mathrm{Fil}^\cdot \mathcal{E}_K}) \;\middle|\; \begin{array}{l} \cdot\ (\mathcal{E}_0, \varphi_0) = F\text{-isocrystal on } (X_0/W)_{\mathrm{crys}} \\ \cdot\ \mathcal{E}_K := \mathcal{E}_0(X_K) \text{ w/ } \mathrm{Fil}^\cdot \mathcal{E}_K \\ \text{and } \nabla : \mathcal{E}_K \to \mathcal{E}_K \otimes \Omega^1_{X_K/K} \\ \quad \text{satisfying Griffith transversality} \end{array} \right\}. \end{array}$$

Take $C = \hat{\bar{K}}$.

$$\rightsquigarrow \quad \begin{array}{ccc} H^i_{\acute{e}t}(X_{C,\acute{e}t}, \mathbb{L}) & \longleftrightarrow & H^i_{\mathrm{cris}}((X_0/W)_{\mathrm{cris}}, \mathcal{E}_0) \\ & \nwarrow\!\!\searrow \quad \swarrow\!\!\nearrow & \\ & H^i(X_{\Delta}, \mathcal{E}_{\Delta}) & \end{array} \qquad A_{\mathrm{inf}} = W(\mathcal{O}_C^\flat).$$

prism - ét comparison, prism - cris comparison $\Big\} \Rightarrow$ ét-cris Comparison.

$(\mathcal{E}_{\Delta}, \varphi_{\mathcal{E}_{\Delta}}) \in \mathrm{Vect}^{\varphi, \mathrm{an}}(X_{\Delta})$.

abs prism site of B-S.

<u>§2 Prismatic preliminaries</u>

· A bounded prism is a pair (A, I).

· A $\mathbb{Z}_p$-alg & $\delta : A \to A$ s.t. $\varphi(x) = x^p + p\delta(x)$ is an endo.

· $I \subseteq A$ loc generated by $d \in A$ s.t. $\delta(d) \in A^\times$.

· A/I has bounded p-torsion ($A/I[p^\infty] = A/I[p^n]$ for some n.) and A is complete for (p, I)-adic top.

E.g. (1) $\mathcal{O}_C^\flat := \varprojlim_{x \mapsto x^p} (\mathcal{O}_C/p)$

$A_{\mathrm{inf}} := W(\mathcal{O}_C^\flat) \circlearrowleft \varphi \leftrightarrow \delta$ s.t. $\delta([x]) = 0$,

$\varepsilon := (1, \zeta_p, \zeta_{p^2}, \cdots) \in \mathcal{O}_C^\flat$,

$\mu := [\varepsilon] - 1 \in A_{\mathrm{inf}}$, $\tilde{\xi} := \varphi(\mu)/\mu = \sum_{i=0}^{p-1} [\varepsilon]^i$.

$\rightsquigarrow (A_{\mathrm{inf}}, (\tilde{\xi}))$ is a prism.

(2) $A = W[[u]] \circlearrowleft \delta(u) = 0 \iff \varphi(u) = u^p$.

$E(u) \in A$ Eisenstein poly, $(A, E(u))$ is a prism.

<u>Def'n</u> (Koshikawa) A bounded prelog prism is a tuple $(A, I, \alpha: M \to A, \delta_{\log})$.

- (A, I) is a bounded prism.
- $\alpha: M \to A$ prelog structure.
- $\delta_{\log}: M \to A$ s.t. $\delta_{\log}(e) = 0$, $\delta(\alpha(m)) = \alpha(m)^p \delta_{\log}(m)$,

$$\delta_{\log}(mm') = \delta_{\log}(m) + \delta_{\log}(m') + p\,\delta_{\log}(m)\delta_{\log}(m').$$

A bounded prelog prism is log-prism if $M \xrightarrow{\sim} \Gamma(\mathrm{Spf}(A), \underline{M}^a)$.

E.g. (1) $(A_{\mathrm{inf}}, (\tilde{\xi}))$, $\alpha: M = \mathcal{O}_C^\flat \setminus \{0\} \longrightarrow A_{\mathrm{inf}}$ $\quad \delta_{\log}(x) = 0$.

$$x \longmapsto [x].$$

(2) $(W[[u]], (E(u)))$, $M = \mathbb{N} \longrightarrow W[[u]]$, $\delta_{\log} = 0$.

$$1 \longmapsto u$$

Let $X/\mathrm{Spf}\,\mathcal{O}_K$ be a semistable log formal sch.

$$M_X := \mathcal{O}_{X_{et}} \cap \mathcal{O}_{X_{et}}[\tfrac{1}{p}]^\times \longrightarrow \mathcal{O}_{X_{et}}.$$

<u>Def'n</u> $X_{\Delta}^{\log}$: site, obj: (A, I, M_A) int bounded log prism $+ (\mathrm{Spf}(A/I), M_A)^a \longrightarrow X^{\log}$.

Covering : flat covers $(A, I, M_A) \longrightarrow (B, J, M_B)$.

- $\mathcal{O}_\Delta : (A, I, M_A) \longmapsto A$.

$$\rightsquigarrow \quad \mathrm{Vect}(X_\Delta^{\log}, \mathcal{O}_\Delta)^{\varphi=1} \cong \varprojlim_{(A,I,M_A) \in \mathrm{Ob}(X_\Delta^{\log})} \mathrm{Vect}(A)^{\varphi=1}.$$

$$\mathrm{Vect}(A)^{\varphi=1} := \left\{ \left(\underset{\substack{\uparrow \\ \text{fin proj } A\text{-mod}}}{\mathcal{E}_A}, \varphi_{\mathcal{E}_A}\right) \,\middle|\, \varphi_{\mathcal{E}_A} : (A \otimes_{\varphi_A} \mathcal{E}_A)[\tfrac{1}{I}] \xrightarrow{\sim} \mathcal{E}_A[\tfrac{1}{I}] \right\}.$$

$$\mathrm{Vect}(X_\Delta^{\log}, \mathcal{O}_\Delta[\tfrac{1}{I}]_p^\wedge) \cong \varprojlim_{(A,I,M_A) \in \mathrm{Ob}(X_\Delta^{\log})} \mathrm{Vect}(A[\tfrac{1}{I}]_p^\wedge).$$

<u>Thm</u> (Bhatt - Scholze + ε)

$$\mathrm{Vect}(X_\Delta^{\log}, \mathcal{O}_\Delta[\tfrac{1}{I}]_p^\wedge) \simeq \mathrm{Loc}_{\mathbb{Z}_p}(X_{\eta,\mathrm{et}}).$$

So $T : \mathrm{Vect}(X_\Delta^{\log}, \mathcal{O}_\Delta)^{\varphi=1} \longrightarrow \mathrm{Vect}(X_\Delta^{\log}, \mathcal{O}_\Delta[\tfrac{1}{I}]_p^\wedge)^{\varphi=1} \simeq \mathrm{Loc}_{\mathbb{Z}_p}(X_{\eta,\mathrm{et}})$.

$$\begin{array}{ccccc}
 & X^\diamond_{C,v} & \xrightarrow{\quad\alpha\quad} & X_{\mathcal{O}_C,\mathrm{qsyn}} & \longleftarrow X^{\log}_{\mathcal{O}_C,\Delta} \\
 & {\scriptstyle\beta}\big\downarrow & & \big\downarrow{\scriptstyle u} & \\
\mathbb{L} - & X_{C,\mathrm{proet}} & \xrightarrow{\quad\nu\quad} & X_{\mathcal{O}_C,\mathrm{et}} &
\end{array}$$

<u>Thm</u> $\forall (\mathcal{E}, \phi_\mathcal{E}) \in \mathrm{Vect}(X_\Delta^{\log}, \mathcal{O}_\Delta)^{\varphi=1}$, $T(\mathcal{E}) \in \mathrm{Loc}_{\mathbb{Z}_p}(X_{C,\mathrm{et}})$.

$\rightsquigarrow$ Then $\exists$ a canonical isom

$$R\nu_*(T(\mathcal{E}) \otimes_{\mathbb{Z}_p} \mathbb{A}_{\inf}[\tfrac{1}{\mu}]) \xrightarrow{\sim} Ru_*(\mathcal{E}[\tfrac{1}{\mu}]).$$

$\mathbb{A}_{\inf}$: sheaf version of $A_{\inf}$ on $X_{C,\mathrm{proet}}$. $\mu = [\varepsilon] - 1 \in A_{\inf}$.

<u>Cor</u> Assume X proper. Then $\exists$ a canonical isom

$$R\Gamma(X_{C,\mathrm{et}}, T(\mathcal{E})) \otimes A_{\inf}[\tfrac{1}{\mu}] \xrightarrow{\sim} R\Gamma(X_\Delta^{\log}, \mathcal{E}[\tfrac{1}{\mu}]).$$

$$\begin{array}{ccc}
\rightsquigarrow \quad L\eta_\mu Ru_* \alpha_* \alpha^*(\mathcal{E}_{\mathrm{qsyn}}) & \xrightarrow{\ \cong\ } & R\nu_* R\beta_*(\beta^* \mathbb{L} \otimes \mathbb{A}_{\inf}[\tfrac{1}{\mu}]) \\
\big\uparrow{\scriptstyle\cong} & & \\
Ru_*(\mathcal{E}_{\mathrm{qsyn}}). & &
\end{array}$$