Lang Conjecture for finite covers of abelian varieties

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Geometric Bombieri-Lang conjecture (jt. with Junyi Xie).

§1 Bombieri-Lang over number fields

- Mordell-Lang conj / Faltings's thm:

  K number field. X/K (sm proj, geom conn) curve, $g(X)>1$.

  $\Rightarrow \# X(K) < \infty$.

- Higher dim'l generalizations

  $g>1$. two versions: (1) hyperbolic: $X(K)$ finite.

  (2) general type: $X(K)$ not Zar dense.

Def'n (1) $X/\mathbb{C}$ proj var. An entire curve is a non-const holo map $\phi: \mathbb{C} \to X(\mathbb{C})$.

The analytic special set $Sp_{an}(X)$ = Zar closure of union of all entire curves.

Say X is (Brody) hyperbolic if $Sp_{an}(X) = \phi$.

(2) X/K proj. The algebraic special set

$Sp_{alg}(X)$ = Zar closure of union of images of non-const rat map

$A \dashrightarrow X_{\bar{K}}$, for abelian var $A/\bar{K}$.

Say X is alg. hyperbolic if $Sp_{alg}(X) = \phi$.

Green-Griffiths-Lang conj $X/\mathbb{C}$ proj. Then

(1) $Sp_{alg}(X) = Sp_{an}(X)$

(2) If X is of general type, then $Sp_{an}(X) \neq X$.

Bombieri-Lang Conj. X/K proj / number field.

Then $(X \setminus \mathrm{Sp}_{alg}(X))(K)$ finite.

Known (i) $\dim X = 1$ $(g > 1)$: Faltings.

(ii) X subvar of abelian var : Faltings.

Thm $X/\mathbb{C}$ with fin morphism $f: X \to A$ to abelian var $A/\mathbb{C}$.

$\Rightarrow$ GGL conj holds.

(Due to : Ueno, Kawamata, Yamanoi.)

§2 Geometric Bombieri-Lang

Conj (GBL) k alg closed, char $k = 0$ (e.g. $k = \mathbb{C}$).

K/k fin gen ext'n. X/K proj var.

(1) There are only fin many closed subvars $Z_i \subset X$

which are (birationally) constant with $Z_i \not\subset \mathrm{Sp}_{alg}(X)$.

(i.e. birat. to base change of a var via $k \to K$ with dim > 0)

Denote $\widetilde{\mathrm{Sp}}_{alg}(X) = \mathrm{Sp}_{alg}(X) \cup (\bigcup_i Z_i)$

(2) $(X \setminus \widetilde{\mathrm{Sp}}_{alg}(X))(K)$ finite.

Known (i) $\dim X = 1$, by Manin - Graueit.

(ii) Subvar of abel var. by Raynaud.

(iii) X sm with ample Ω_X^1 by Noguchi.

Main Thm $X/K/k$ proj.

Assume X fin over an abel var $/K$.

Then GBL holds for X if

either (a) X hyperbolic.

or (b) The (K/k)-trace of A is 0

(i.e. any homomorph $B_{0,\bar{K}} \to A$ for B_0/k abel var is 0).

§3 Idea of proof

(*) To construct entire curves on $\mathcal{X}$ (int model) from inf seq of $X(K)$.

<u>Assume</u> $k = \mathbb{C}$, $K = \mathbb{C}(B)$, $B/\mathbb{C}$ proj curve. X hyperbolic.

$\pi : \mathcal{X} \longrightarrow B$ int model (regular, flat, int.)

Also ass: $\{x_n\} \subseteq X(K)$ inf seq (want a contradiction).

L ample on $X \rightsquigarrow \mathcal{L}$ rel ample on $\mathcal{X}$.

ω Kahler form rep. $c_1(\mathcal{L})$ on $\mathcal{X}$.

<u>Weil ht</u> $h_{\mathcal{L}} : X(K) \longrightarrow \mathbb{R}$, $x \longmapsto \deg_{\mathcal{L}}(\tilde{x}) = \int_B \tilde{x}^* \omega$.

where $\tilde{x} : B \to \mathcal{X}$ corr to x.

<u>Partial ht</u> $D \subset B$ disc. $h_{(D,\omega)} : X(K) \to \mathbb{R}$, $x \longmapsto \int_D \tilde{x}^* \omega$.

<u>Conj</u> (non-deg) If X doesn't contain any rat curve,

then $\forall \{x_n\} \subset X(K)$ with $h_{\mathcal{L}}(x_n) \to \infty$,

have $h_{(D,\omega)}(x_n) \longrightarrow \infty$.

<u>Step 1</u> (X fin / A hyperbolic) If $h_{\mathcal{L}}(x_n)$ is bounded, then X is constant.

(Assume $\{x_n\}$ Zar dense in X.) (Due to Noguchi).

$\rightsquigarrow \{\tilde{x}_n\}$ bounded family $\Rightarrow$

$$\begin{tikzcd} B \times_{\mathbb{C}} \underline{\mathrm{Hilb}} \arrow[r, "\text{dominant}"] \arrow[dr] & \mathcal{X} \arrow[d] \\ & B \end{tikzcd}$$

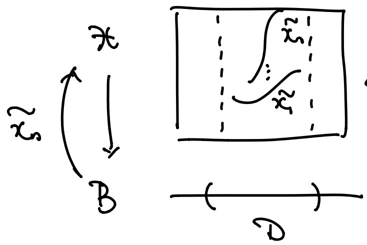
<u>Step 2</u> If $h_{\mathcal{L}}(x_n) \to \infty$, then $h_{(D,\omega)}(x_n) \to \infty$ (non-deg conj.)

Idea: apply Mok's Betti form $\omega = \omega_{\mathrm{Betti}}$ on $\mathcal{X}_{B^\circ}$.

$h_{(D,\omega)} : (A(K)/(\text{trace part}))_{\mathbb{R}} \longrightarrow \mathbb{R}$ pos. def'te quadratic form.

<u>Step 3</u> Apply Bordy's lemma:

("discs" in $\mathcal{X}$ converges to entire curves in $\mathcal{X}$.)



$\leftarrow \int_D \tilde{x}_n^* \omega \to \infty$, $\tilde{x}_n$ gets to be "∞".

$\rightsquigarrow$ eventually: $\{\tilde{x}_n : D \to \mathcal{X}\}_n$.

$\rightsquigarrow$ Converges to an entire curve on $\mathcal{X}$ (in a fiber of $\mathcal{X}/B$).

$\Rightarrow$ a contradiction to hyperbolicity.

<u>Rmk</u> Philosophy via Vojta's dictionary:

Nevanlinna theory $\longleftrightarrow$ Diophantine geom.

(an entire curve) $\longleftrightarrow$ (an infin seq of rat pts).