p-adic height of arithmetic diagonal cycles on unitary Shimura varieties

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(Joint with D. Disegni).

$$\text{Alg cycles} \longleftrightarrow \text{L-functions}$$

<u>Conj</u> (B-SD, Beilinson, Bloch : higher dim ver)

X sm proj / F = number field.

$Ch^i(X)$ = alg cycles of codim i / rat'l equiv.

w/ $\mathbb{Q}$-coeff.

Consider Hasse-Weil L-function $H^i(X_{\bar F}, \mathbb{Q}_p) \circlearrowleft Gal(\bar F/F)$.

$\rightsquigarrow L(H^i(X), s) = \prod_v \det(1 - \mathrm{Frob}_v q_v^{-s} | H^i)^{-1}$.

zero set $\updownarrow$ essentially char poly of Frob.

$\{\# X(\mathbb{F}_{q_v^n}) | n\}$.

$\rightsquigarrow Ch^i(X) \xrightarrow{cl} H^i_{Betti}(X(\mathbb{C}), \mathbb{Q})$. cycle class map

$\cup\!\!|$

$Ch^i(X)_0$ subgp of deg 0 divisors.

<u>Then</u> $\dim_{\mathbb{Q}}(Ch^i(X)_0) = \mathrm{ord}_{s=i} L(H^{2i-1}_{et}(X), s)$.

<u>Example</u> X curve, $\mathrm{rank}\, Jac_X(F) \overset{BSD}{=} \mathrm{ord}\, L(Jac_X, s)$.
$(g=1)$

higher rank $\leftrightarrow$ more rat'l pt on X.

Can look at $\{\# X(\mathbb{F}_p)\}_p$:

$$\mathrm{rank}\, X(F) = 0 \iff \prod_p \frac{\# X(\mathbb{F}_p)}{p} < \infty.$$

<u>Fact</u> B-SD over function field $\underset{\text{Tate}}{\overset{\text{Artin}}{\Longleftrightarrow}}$ Tate conj for (elliptic) surfaces / $\mathbb{F}_q$.
(for $C/\mathbb{F}_q$)

$$\begin{array}{ccc} X & \longleftrightarrow & \mathfrak{X} \text{ int model} \\ \downarrow & & \downarrow \\ \mathrm{Spec}\, F & & C \;/\mathbb{F}_q. \end{array}$$

Shimura variety case:

<u>Arith GGP Conj</u> (for unitary gp)

$$H \hookrightarrow G \text{ unitary} \qquad \qquad \mathrm{Sh}_H \hookrightarrow \mathrm{Sh}_G$$

F/F_0 = CM ext'n.

$$G = U(n-1,1) \times U(n,1)$$

$$H = U(n-1,1) \hookrightarrow G$$

$$U(n) \longrightarrow U(n+1), \quad g \longmapsto \begin{pmatrix} g & 0 \\ 0 & 1 \end{pmatrix}.$$

$\mathrm{Sh}_G(\mathbb{C})$ = Ball quotient, $\dim = n-1+n = 2n-1$

$\dim \mathrm{Sh}_H = n-1$.

$$\rightsquigarrow H^*(\mathrm{Sh}_G) = \bigoplus_{\pi \in \Pi(G(\mathbb{A}))} \pi \boxtimes \rho_\pi \qquad (\pi: \text{cusp autom rep'n of } G(\mathbb{A})).$$

Hecke × Gal_F $\circlearrowleft$; Hecke, Galois

<u>Fact</u> $\dim_{\mathbb{Q}} \mathrm{Ch}^i(\mathrm{Sh}_G)_\pi = \operatorname{ord}_{s=1/2} L(\rho_\pi, s)$.

Conj (AGGP) $\mathrm{Sh}_{H,\pi} \neq 0 \iff \operatorname{ord}_{s=1/2} L(\pi,s) = 1$.

p-adic height at π up to local GGP

<u>Punchline</u> p-adic L-function $\rightsquigarrow X/F(\mu_{p^n})$ $(n \geq 1)$.

<u>Thm</u> (D-Z, Mihatsch-Z, Zhiyu Zhang).

If $\mathrm{Sh}_{G,K}$ has good or strictly semistable reduction at all inert places of F/F_0,

where F/F_0 quad ext'n unramified everywhere,

and π is stable & ordinary at p,

<u>then</u> $\operatorname{ord} L_p(\pi, s) = 1 \Rightarrow \mathrm{Sh}_{H,\pi} \neq 0$.

<u>Strategy</u> X/F, $i+j = \dim_F X + 1$.

$$\mathrm{Ch}^i_0(X) \times \mathrm{Ch}^j_0(X) \longrightarrow \mathbb{R} \text{ or } \mathbb{Q}_p$$

$$\langle z_1, z_2 \rangle = \sum_v \langle z_1, z_2 \rangle_v, \quad |z_1| \cap |z_2| = \phi.$$

For v non-arch,

$$\begin{array}{ll} \mathcal{X}_v & \langle z_1, z_2\rangle_v = \chi(\mathcal{O}_{\tilde{z}_1} \overset{\mathbb{L}}{\otimes} \mathcal{O}_{\tilde{z}_2}) \cdot \log q_{v}. \\ \big\downarrow & v \text{ arch}, \ \langle z_1, z_2\rangle_\infty = \int_{z_2(\mathbb{C})} \underset{\substack{\uparrow \\ \text{Green current.}}}{g_{z_1}} \\ \mathcal{O}_{F_v} & \end{array}$$

Consider p-adic Abel-Jacobi :

$$\begin{tikzcd} \mathrm{Ch}_0^i(X) \times \mathrm{Ch}_0^j(X) \arrow[d] \arrow[dr] & \\ H_f^1(F, H^{2i-1}(X)) \times H_f^1(F, H^{2j-1}(X)) \arrow[r] & \mathbb{Q}_p \end{tikzcd}$$

<u>Cor</u> $\operatorname{ord} L_p(\pi, S) = 1 \Rightarrow \dim H_f^1 = 1$,

$\operatorname{ord} L_p(\pi, S) = 0 \Rightarrow \dim H_f^1 = 0$.