4.12 Averaging functors in Fargues' program for GL_n

Notation E non-arch local field, $\mathbb{F}_q$ res. field, $\varpi \in E$

$C = \hat{\bar{E}}$, $\breve{E} = \widehat{E^{ur}}$, $k = \overline{\mathbb{F}_q}$

$$\mathrm{Bun}_n : \mathrm{Perf}_k \longrightarrow \mathrm{Gpd}$$
$$S \longmapsto \{\text{v.b. of rk } n / X_{S,E}\}$$

$|\mathrm{Bun}_n| = B(G) = G(\breve{E})/_{\sigma\text{-conj}}$

$\forall b \in B(G)$, $j_b : \mathrm{Bun}_n^b \hookrightarrow \mathrm{Bun}_n$

$$j : \mathrm{Bun}_n^{ss} = \bigsqcup_{b \in B(G)_{basic}} \mathrm{Bun}_n^b \hookrightarrow B$$

$\mathrm{Div}' = \mathrm{Spd}(\breve{E})/_{\varphi^{\mathbb{Z}}}$, $\pi_1(\mathrm{Div}') \simeq W_E$

FS $D_{lis}(\mathrm{Bun}_n, L)$ $\forall$ $\mathbb{Z}_\ell$-alg L, $\ell \neq p$

($\overset{\shortparallel}{D_{\acute{e}t}}(\mathrm{Bun}_n, L)$ if L is torsion)

$\forall$ finite set I,

$$T^I : \mathrm{Rep}(\hat{G})^I \times D_{lis}(\mathrm{Bun}_G, L) \longrightarrow D_{\blacksquare}(\mathrm{Bun}_n \times (\mathrm{Div}')^I, L)$$
$$(V, \mathcal{F}) \longmapsto T_V^I(\mathcal{F})$$

$\mathbb{L}$ be an n dim'l rep'n of W_E, i.e.

$$W_E \longrightarrow \hat{G}(L)$$

$\forall$ alg rep'n $r_V : \hat{G}_L \longrightarrow GL(V)$

$$\rightsquigarrow r_{V,*}(\mathbb{L}) : W_E \longrightarrow \hat{G}(L) \longrightarrow GL(V)$$

Def $\mathcal{F} \in D_{lis}(\mathrm{Bun}_n, \overline{\mathbb{Q}}_\ell)$ is called a Hecke eigensheaf with eigenvalue $\mathbb{L}$ if $\forall I$, $(V_i)_{i \in I} \in \mathrm{Rep}(\hat{G})^I$

$$\exists\ \eta_{(V_i)_{i\in I}} : T^I_{(V_i)_{i \in I}}(\mathcal{F}) \xrightarrow{\sim} \mathcal{F} \boxtimes \left(\boxtimes_{i \in I} r_{V_i,*}(\mathbb{L})\right)$$

natural in I, compatible with composition & exterior prod.

Conj (Fargues)

$\forall$ irr $\overline{\mathbb{Q}}_\ell$-rep'n $\mathbb{L}$ of W_E of dim n, $\exists$ $\mathrm{Aut}_{\mathbb{L}} \in D_{\mathrm{lis}}(\mathrm{Bun}_G, \overline{\mathbb{Q}}_\ell)$ s.t.

(1) $\mathrm{Aut}_{\mathbb{L}}$ is a Hecke eigensheaf with eigenvalue $\mathbb{L}$

(2) $\mathrm{Aut}_{\mathbb{L}} \simeq \bigoplus_{b \in B(G)_{\mathrm{basic}}} j_{b,!}(\mathcal{F}_{LL_b(\mathbb{L})})$

$\mathcal{F}_{LL_b(\mathbb{L})} \in D_{\mathrm{lis}}(\mathrm{Bun}_n^b, \overline{\mathbb{Q}}_\ell) \simeq D(\mathrm{Rep}^\infty_{\overline{\mathbb{Q}}_\ell}(G_b(E)))$ corresponding to $LL_b(\mathbb{L})$ via JL cor.

$\mathrm{Bun}_n^b = [*/\underline{G_b(E)}]$

Main thm of [AL] Fargues' conj. is true!

Main tool: Averaging functor

$$\mathrm{Av}_{\mathbb{L}} : D_{\mathrm{lis}}(\mathrm{Bun}_G, \overline{\mathbb{Q}}_\ell) \longrightarrow D_{\mathrm{lis}}(\mathrm{Bun}_G, \overline{\mathbb{Q}}_\ell)$$

Constant term

$P \subset G = GL_n$ standard parabolic, $M \subset P \twoheadrightarrow M$

$$\mathrm{Bun}_n \xleftarrow{p} \mathrm{Bun}_P \xrightarrow{q} \mathrm{Bun}_M$$

Lem (1) p is representable in loc. spatial diamonds

compactifiable & loc of dim $\mathrm{trg}(p) < \infty$

(2) q is coh. sm.

Pf (1) $S \in \mathrm{Perf}_k$, $\mathcal{E} \in \mathrm{Bun}_n(S)$

$$\mathrm{Bun}_P \times_{\mathrm{Bun}_n} S = \{\text{red. of } \mathcal{E} \text{ to } P\} = \mathcal{M}_Z$$

$$Z := \mathcal{E}/P \longrightarrow X_{S,E}$$

[FS IV.4.2] $\mathcal{M}_Z$ is rep in loc. spatial diamond

(2) WLOG $M = GL_{n_1} \times GL_{n_2}$

Given $\mathcal{F} \in \mathrm{Bun}_M(S)$, $\mathcal{F} = \overset{n_1}{\mathcal{E}_1} \times \overset{n_2}{\mathcal{E}_2}$

$$\mathrm{Bun}_P \times_{\mathrm{Bun}_M} S = \mathcal{E}xt^1(\mathcal{E}_2, \mathcal{E}_1) : \mathrm{Perf}_S \longrightarrow \mathrm{Gpd}$$

$$T \longmapsto \{0 \to \mathcal{E}_{1,T} \to \mathcal{E} \to \mathcal{E}_{2,T} \to 0\}$$

WLOG S affinoid, consider surjection

$$\mathcal{O}_{X_S}(-d)^N \twoheadrightarrow \mathcal{E}_2^\vee$$

$$\rightsquigarrow \quad 0 \to \mathcal{E}_2 \to \mathcal{O}_{X_S}(d)^N \to \mathcal{G} \to 0$$

Enlarging d, s.t. $\mathcal{E}_1^\vee \otimes \mathcal{O}_{X_S}(d)^N$, $\mathcal{E}_1^\vee \otimes \mathcal{G}$ have positive slopes.

$X_1 := \underline{\mathrm{Hom}}(\mathcal{E}_1, \mathcal{O}_{X_S}(d)^N)$, $X_2 = \underline{\mathrm{Hom}}(\mathcal{E}, \mathcal{G})$

$$\mathcal{E}xt^1(\mathcal{E}_1, \mathcal{E}_2) = X_2/X_1 \qquad \square$$

Λ = torsion $\mathbb{Z}_l[\sqrt{q}]$-alg

<u>Def</u> $$CT_{P,!} : D_{\acute{e}t}(\mathrm{Bun}_n, \Lambda) \longrightarrow D_{\acute{e}t}(\mathrm{Bun}_M, \Lambda)$$
$$\mathcal{F} \longmapsto q_! p^* \mathcal{F}$$

$$\mathrm{Eis}_{P,*} : D_{\acute{e}t}(\mathrm{Bun}_M, \Lambda) \longrightarrow D_{\acute{e}t}(\mathrm{Bun}_n, \Lambda)$$
$$\mathcal{G} \longmapsto p_* q^! \mathcal{G}$$

$\rightsquigarrow (CT_{P,!}, \mathrm{Eis}_{P,*})$ adjoint pair

<u>Def</u> An $\mathcal{G} \in D_{\acute{e}t}(\mathrm{Bun}_n, \Lambda)$ is <u>cuspidal</u> if

$\forall P \subset G$ standard parabolic,

$$CT_{P,!}(\mathcal{G}) = 0$$

$\rightsquigarrow D_{\acute{e}t, cusp}(\mathrm{Bun}_n, \Lambda)$

<u>Twisting</u>

$$\begin{tikzcd} & \mathrm{Bun}_P \arrow[dl, "p"'] \arrow[dr, "q"] \arrow[dd, "\det_P"] & \\ \mathrm{Bun}_n \arrow[dr, "\det"'] & & \mathrm{Bun}_M \arrow[dl, "\det_M"] \\ & \mathrm{Bun}_1 & \end{tikzcd}$$

<u>lem</u> $\forall \mathcal{G} \in D_{\acute{e}t}(\mathrm{Bun}_1)$, $\mathcal{F} \in D_{\acute{e}t}(\mathrm{Bun}_n)$

$$CT_{P,!}(\mathcal{F} \otimes^{\mathbb{L}}_{\Lambda} \det{}^* \mathcal{G}) \xrightarrow{\sim} CT_{P,!}(\mathcal{F}) \otimes^{\mathbb{L}}_{\Lambda} \det{}_M^* \mathcal{G} \qquad \square$$

<u>lem</u> Any $\mathcal{F} \in D_{\acute{e}t, cusp}(\mathrm{Bun}_n)$ is supported on Bun_n^{ss}

Pf $b \in B(G)$ non s.s. , $\underset{\overset{\cap}{P}}{M}$ standard Levi attached to b

$b \in B(M)$

$\mathrm{Bun}_P^b := \mathrm{Bun}_P \times_{\mathrm{Bun}_M} \mathrm{Bun}_M^b$

Then $D_{\acute{e}t}(\mathrm{Bun}_P^b) \simeq D_{\acute{e}t}(\underbrace{\mathrm{Bun}_M^b}_{\overset{\shortparallel}{[*/G_b(E)]}})$

(Since fiber of $\mathrm{Bun}_P^b \longrightarrow \mathrm{Bun}_M^b$ is a positive BC space)

$\Rightarrow j_{b,*}(D_{\acute{e}t}([*/\underline{G_b(E)}], \Lambda)) \subset \mathrm{Im}(\mathrm{Eis}_{P,*})$

$\mathcal{F} \in D_{\acute{e}t,cusp}(\mathrm{Bun}_n)$, $\mathcal{G} \in D_{\acute{e}t}([*/\underline{G_b(E)}])$

$j_{b,*}\mathcal{G} = \mathrm{Eis}_{P,*}(\mathcal{G}')$

$\mathrm{Hom}(j_b^*\mathcal{F}, \mathcal{G}) \simeq \mathrm{Hom}(\mathcal{F}, j_{b,*}\mathcal{G}) \simeq \mathrm{Hom}(\mathcal{F}, \mathrm{Eis}_{P,*}(\mathcal{G}')) \simeq \mathrm{Hom}(CT_{P,!}\mathcal{F}, \mathcal{G}') = 0$

Def $b \in B(G)_{basic}$, irr. $\pi \in \mathrm{Rep}_\Lambda^\infty(G_b(E))$

(1) π is called cuspidal if

$J_P(\pi) = 0$, $\forall$ proper parabolic $P \subset G_b$

$J_P : \mathrm{Rep}_\Lambda^\infty(G_b(E)) \longrightarrow \mathrm{Rep}_\Lambda^\infty(M)$

$\Leftrightarrow$ π is not a subrep of $\mathrm{Ind}_P^{G_b}(\tau)$, $\tau \in \mathrm{Rep}_\Lambda^\infty(M)$

π is called supercuspidal if

π is not a subquotient of $\mathrm{Ind}_P^{G_b}(\tau)$, $\tau \in \mathrm{Rep}_\Lambda^\infty(M)$

(When char $\Lambda = 0$, cusp. = supercusp.)

(2) π is geometrically cuspidal if

$j_{b,!}(\mathcal{F}_\pi) \in D_{\acute{e}t,cusp}(\mathrm{Bun}_n)$

Prop (1) π is geo. cusp. $\Rightarrow$ π is cusp.

(2) $n = 2$, $G_b \simeq GL_2$ (i.e. $\mu(b) \in 2\mathbb{Z}$)

π is geo. cusp. $\Leftrightarrow$ supercusp.

Proof (1) $P_b \subset G_b$ parabolic with Levi M_b

$\exists$ parabolic $P \subset G$ with Levi M s.t. $\quad$ (G quasi-split)

$$\mathrm{Bun}_P \simeq \mathrm{Bun}_{P_b} \quad , \quad \mathrm{Bun}_M \simeq \mathrm{Bun}_{M_b}$$

$c \in \mathrm{Bun}_M$ cor. to trivial M_b-torsor

$$[*/P_b] \simeq \mathrm{Bun}_P^c := \mathrm{Bun}_P^c := \mathrm{Bun}_P \times_{\mathrm{Bun}_M} \mathrm{Bun}_M^c \longrightarrow \mathrm{Bun}_n$$

has image in Bun_n^b

$$\begin{tikzcd}
& {[*/P_b] \simeq \mathrm{Bun}_P^c} \arrow[dl, "p"'] \arrow[dr, "q"] & \\
{[*/G_b(E)]} & & \mathrm{Bun}_M^c = [*/M_b(E)]
\end{tikzcd}$$

$\rightsquigarrow CT_{P,!}(\mathcal{F}_\pi)_c = J_{P_b}(\pi)$

(2) Assume $n=2$, π geo. cusp. $\overset{?}{\Rightarrow}$ supercuspidal

if not, by Vignéras, $q \equiv -1 \mod \ell$

$$\pi = \pi_1 \otimes \chi \circ \det$$

$\cdot\ \pi_1$ unique ∞-dim'l factor of

$$\mathrm{Ind}_B^{GL_2}(1 \boxtimes |\cdot|)$$

???

supercuspidal $\Rightarrow$ geo. cusp.

$CT_{B,!}(\mathcal{F}_\pi) \overset{?}{=} 0$

$$\begin{tikzcd}
& \mathrm{Bun}_2^b \times_{\mathrm{Bun}_2} \mathrm{Bun}_P \subset \mathrm{Bun}_P \arrow[dl] \arrow[d] \arrow[dr] & \\
\mathrm{Bun}_2^b \subset & \mathrm{Bun}_2 & \mathrm{Bun}_1 \times \mathrm{Bun}_1
\end{tikzcd}$$

$\mathcal{F}_\pi \qquad\qquad \mathcal{O}(m) \times \mathcal{O}(d-m)$

$d = \kappa(b) \in 2\mathbb{Z}$

$\Leftrightarrow \forall m, \quad Z_m := \mathrm{Hom}(\mathcal{E}_b, \mathcal{O}(m))$

$$\cup$$

$$Z_m^\circ = \mathrm{Hom}^{\mathrm{surj}}(\mathcal{E}_b, \mathcal{O}(m))$$

$$R\Gamma_{\mathrm{c}}(GL_2(E), R\Gamma_c(\tilde{Z}_m^\circ, \Lambda) \otimes \pi) = 0$$

where $\widetilde{Z}_m^\circ \longrightarrow Z_m^\circ$

$\quad\;\, \|$

$(0 \to \mathcal{O}(d-m) \to \mathcal{E}_b \to \mathcal{O}(m) \to 0)$

<u>Averaging functor</u>

$$\begin{tikzcd} \mathrm{Div}' & \mathrm{Mod}_n' \arrow[l, "\alpha"'] \arrow[dl, "\overleftarrow{h}"'] \arrow[dr, "\overrightarrow{h}"] & \\ \mathrm{Bun}_n & & \mathrm{Bun}_n \end{tikzcd}$$

x, $\mathcal{E}$, $\mathcal{E}'$

$\mathrm{Mod}_n' : S \longmapsto \{(x, \mathcal{E} \hookrightarrow \mathcal{E}') \mid \cdot\ x \in X_S$

$\cdot\ 0 \to \mathcal{E} \to \mathcal{E}' \longrightarrow i_* C_x \longrightarrow 0\ \}$

<u>Def</u> $\mathbb{L} \in D_{\acute{e}t}(\mathrm{Div}', \Lambda)$

$\mathrm{Av}_{\mathbb{L},n} : D_{\acute{e}t}(\mathrm{Bun}_n) \longrightarrow D_{\acute{e}t}(\mathrm{Bun}_n)$

$\mathcal{F} \longmapsto \overrightarrow{h}_!(\overleftarrow{h}^*(\mathcal{F}) \otimes^{\mathbb{L}}_{\Lambda} \alpha^* \mathbb{L}(\tfrac{n-1}{2})[n-1])$

<u>Prop</u> $P = \begin{pmatrix} \llcorner^{n_1} & * \\ & \llcorner_{n_2} \end{pmatrix} \subset G = GL_n$

$M = GL_{n_1} \times GL_{n_2}$

$\exists\ (\mathrm{Av}_{\mathbb{L}(\frac{n_2}{2})[n_2], n_1} \times \mathrm{Id}_{\mathrm{Bun}_{n_2}}) \circ CT_{P,!} \longrightarrow CT_{P,!} \circ \mathrm{Av}_{\mathbb{L},n} \longrightarrow (\mathrm{Id}_{\mathrm{Bun}_{n_1}} \times \mathrm{Av}_{\mathbb{L}(\frac{n_1}{2})[n_1], n_2}) \circ CT_{P,!} \xrightarrow{+1}$ dist. triangle.

<u>Proof</u>

$$\begin{tikzcd} & & Y \arrow[dl, "p_Y"'] \arrow[dr, "q_Y"] \arrow[d, phantom, "\wedge"] & & \\ & \mathrm{Mod}_n' \arrow[dl, "\overleftarrow{h}"'] \arrow[dr, "\overrightarrow{h}"] & & \mathrm{Bun}_P \arrow[dl, "p"'] \arrow[dr, "q"] & \\ \mathrm{Bun}_n & & \mathrm{Bun}_n & & \mathrm{Bun}_M \end{tikzcd}$$

$Y = \{x,\ 0 \to \underset{n_1}{\mathcal{E}_1'} \to \overset{\mathcal{E}}{\underset{\cup}{}}\mathcal{E}' \to \underset{n_2}{\mathcal{E}_2'} \to 0$

$U \overset{j}{\hookrightarrow} Y$ $\qquad$, $Z = Y \setminus U$

$\|$

$\{(x, 0 \to \mathcal{E}_1' \to \mathcal{E}' \to \mathcal{E}_2' \to 0) \mid \mathcal{E} \to \mathcal{E}' \to \mathcal{E}_2'$ surjective $\}$

$\mathcal{E}$ ↪ $\mathcal{E}'$

$CT_{P,!}\,\mathrm{Av}_{\mathbb{L},n}(\mathcal{F}) = (q \circ q_Y)_!((p_Y^* \circ \overleftarrow{h}^* \mathcal{F}) \otimes \mathbb{L}_Y)$

$\underbrace{j_! j^* \mathbb{L}_Y}_{\mathbb{L}_U} \to \mathbb{L}_Y \to i_* i^* \mathbb{L}_Y \xrightarrow{+1}$

<u>Claim</u> $(q \circ q_Y)_!((p_Y^* \circ \overleftarrow{h}^*)(\mathcal{F}) \otimes \mathbb{L}_U) = (\mathrm{Av} \times \mathrm{Id}_{\mathrm{Bun}_{n_2}}) \circ CT_{P!}$

<u>Key</u>

$$\begin{tikzcd} 0 \arrow[r] & \mathcal{E}_1' \arrow[r] & \mathcal{E}' \arrow[r] & \mathcal{E}_2' \arrow[r] & 0 \\ 0 \arrow[r] & \mathcal{E}_1 \arrow[r] & \mathcal{E} \arrow[r] \arrow[u, hook] & \mathcal{E}_2 \arrow[r] \arrow[u, equal] & 0 \end{tikzcd}$$

$\rightsquigarrow (x, \mathcal{E}_1 \hookrightarrow \mathcal{E}_1') \in \mathrm{Mod}_{n_1}'$

The correspondence by $\mathbb{L}_U$ is identified with

$$\begin{tikzcd} & & U \arrow[dl] \arrow[dr] \arrow[d, phantom, "\wedge"] & & \\ & \mathrm{Bump} \arrow[dl] \arrow[dr] & & \mathrm{Mod}'_{n_1} \times \mathrm{Bun}_{n_2} \arrow[dl] \arrow[dr] & \\ \mathrm{Bun}_n & & \mathrm{Bun}_{n_1} \times \mathrm{Bun}_{n_2} & & \mathrm{Bun}_{n_1} \times \mathrm{Bun}_{n_2} \end{tikzcd}$$

Similar over Z

<u>Prop</u> $\chi : W_E \longrightarrow \Lambda^\times \rightsquigarrow \mathbb{L}_\chi \in D_{\acute{e}t}(\mathrm{Div}', \Lambda)$

Let $\mathcal{G} \in D_{\acute{e}t}(\mathrm{Bun}_1, \Lambda)$ be a Hecke eigensheaf with eigenvalue $\mathbb{L}_\chi$. Then $\forall\, \mathcal{F} \in D_{\acute{e}t}(\mathrm{Bun}_n)$

$\mathrm{Av}_{\mathbb{L}, n}(\mathcal{F} \otimes^{\mathbb{L}}_\Lambda \det^* \mathcal{G}) \simeq \mathrm{Av}_{\mathbb{L} \otimes \mathbb{L}_\chi^\vee}(\mathcal{F}) \otimes \det^* \mathcal{G}$

<u>Thm</u> $\chi : W_E \longrightarrow \Lambda^\times \longrightarrow \mathbb{L}_\chi$

Then $\forall\, \mathcal{F} \in D_{\acute{e}t, cusp}(\mathrm{Bun}_2, \Lambda)$, $\mathrm{Av}_{\mathbb{L}_\chi}(\mathcal{F}) = 0$

<u>Rmk</u> In general, Fargues' conj. suggests if $\mathbb{L}$ is a Λ-l.s. of $\mathrm{rk} \le n$,

then $\mathrm{Av}_{\mathbb{L}, n}(\mathcal{F}) = 0 \quad \forall\, \mathcal{F} \in D_{\acute{e}t, cusp}(\mathrm{Bun}_n)$

<u>Pf</u> WMA $\mathbb{L} = \Lambda$ trivial by previous prop.

Prop 1 $\Rightarrow$ $\mathcal{F}$, $\mathrm{Av}(\mathcal{F}) \in D_{\acute{e}t, cusp}$

$\Rightarrow$ $\mathrm{supp}(\mathcal{F})$, $\mathrm{supp}(\mathrm{Av}\,\mathcal{F}) \subset \mathrm{Bun}_2^{ss}$

sufficies to show $\forall\, b, c$, $\kappa(b) = \kappa(c) + 1$

$\forall\, \mathcal{F} \in D_{\acute{e}t, cusp}$ supported on Bun_2^b

$j_c^* \mathrm{Av}(\mathcal{F}) = 0$

$\forall\, r = 0, \dots, n$, $d \in B(GL_r)$

$Z_{r,d} = \{ (\mathcal{E} \to \mathcal{F} \to \mathcal{E}') \mid \mathcal{E} \simeq \mathcal{E}_b,\ \mathcal{E}' \simeq \mathcal{E}_c,\ \mathcal{F} \simeq \mathcal{E}_d \}$

$$\begin{tikzcd} j_{r,d} : Z_{r,d} \arrow[r] & Z \arrow[dl, "q_1"'] \arrow[dr, "q_2"] & \\ \mathrm{Bun}_n & & \mathrm{Bun}_n \end{tikzcd}$$

$|Z| = \coprod_{r,d} |Z_{r,d}|$

$$\mathrm{Av}_{r,d} : D_{\acute{e}t}(\mathrm{Bun}_n) \longrightarrow D_{\acute{e}t}(\mathrm{Bun}_n)$$

$$\mathcal{G} \longmapsto q_{2!}(q_1^* \mathcal{G} \otimes^{\mathbb{L}}_{\Lambda} j_{c,d!} \Lambda)$$

$$\overline{\mathrm{Av}} : \mathcal{G} \longmapsto q_{2!} q_1^* \mathcal{G}$$

$\mathrm{Av} = \mathrm{Av}_{n,b}$, $Z_{n,d} = \emptyset$ if $d \neq b$

<u>Claim</u> Any of $\mathrm{Av}_{r,d}(\mathcal{F})$, $\overline{\mathrm{Av}}(\mathcal{F})$ vanishes if $r < n$

For $\overline{\mathrm{Av}}$: $Z \times_{\mathrm{Bun}_n^c} k = [G_b(E) \backslash \mathrm{Hom}(\mathcal{E}_b, \mathcal{E}_c)]$

$\uparrow$ positive BC space

$$j_c^* \overline{\mathrm{Av}}(\mathcal{F}) = R\Gamma_c(Z \times_{\mathrm{Bun}_n^c} k, q_1^* \mathcal{F})$$

$$= R\Gamma(G_b(E), \mathcal{F}_b) \overset{?}{=} 0$$

- $\kappa(b) \in 2\mathbb{Z} \Rightarrow R\Gamma(G_b(E), \mathcal{F}_b)$ vanishes as $\mathcal{F}_b$ supercuspidal
- $\kappa(b) \in \mathbb{Z} \backslash 2\mathbb{Z} \Rightarrow \; = 0$ as $G_b(E)$ cpt mod center

$\mathrm{Av}_{r,d}$: $Z_{r,d} \times_{\mathrm{Bun}_n} k = [G_b(E) \backslash \mathrm{Hom}^{surj}(\mathcal{E}_b, \mathcal{E}_d) \overset{\mathcal{G}_d}{\times} \mathrm{Hom}^{inj}(\mathcal{E}_d, \mathcal{E}_c)]$

$$\Big\downarrow \text{fibration}$$

$$[\mathcal{G}_d \backslash \mathrm{Hom}^{inj}(\mathcal{E}_d, \mathcal{E}_c)]$$

$$R\Gamma_c(Z_{r,d} \times_{\mathrm{Bun}_n^c} k, q_1^* \mathcal{F}) \overset{?}{=} 0$$

reduce to show $R\Gamma_c([G_b(E) \backslash \mathrm{Hom}^{surj}(\mathcal{E}_b, \mathcal{E}_d)], q_1^* \mathcal{F}) \overset{?}{=} 0$

$\qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad ||$

$R f_! \, \mathrm{CT}_{P_{r,n-r},!}(\mathcal{F})_{\mathcal{E}_d}$ $\qquad \mathrm{Bun}_r \times \mathrm{Bun}_{n-r} \xrightarrow{f} \mathrm{Bun}_r \ni \mathcal{E}_d$ $\qquad \square$