

4.12 Averaging functors in Fargues' program for GL_n

Notation E non-arch local field, \mathbb{F}_q res. field, $\bar{w} \in E$

$$C = \hat{E}, \quad \tilde{E} = \overline{E^{\text{nr}}}, \quad k = \overline{\mathbb{F}_q}$$

$$\begin{aligned} \text{Bun}_n &: \text{Perf}_k \longrightarrow \text{Gpd} \\ S &\longmapsto \{ \text{v.b. of rk } n / X_{S,E} \} \\ |\text{Bun}_n| &= B(G) = G(\tilde{E}) / \sigma\text{-conj.} \end{aligned}$$

$$\forall b \in B(G), \quad j_b: \text{Bun}_n^b \hookrightarrow \text{Bun}_n$$

$$j: \text{Bun}_n^{\text{ss}} = \bigsqcup_{b \in B(G) \text{ basic}} \text{Bun}_n^b \hookrightarrow B$$

$$\text{Div}^1 = \text{Spd}(\tilde{E}) / \varphi_p \mathbb{Z}, \quad \pi_1(\text{Div}^1) \simeq W_E$$

$$\begin{aligned} \text{FS } D_{\text{is}}(\text{Bun}_n, L) \quad \forall \mathbb{Z}_\ell\text{-alg } L, \ell \neq p \\ \left(D_{\text{ét}}(\text{Bun}_n, L) \text{ if } L \text{ is torsion} \right) \end{aligned}$$

\forall finite set I ,

$$\begin{aligned} T^I: \text{Rep}(\hat{G})^I \times D_{\text{is}}(\text{Bun}_n, L) &\longrightarrow D_{\text{is}}(\text{Bun}_n \times (\text{Div}^1)^I, L) \\ (V, \mathcal{F}) &\longmapsto T_V^I(\mathcal{F}) \end{aligned}$$

\mathbb{L} be an n dim'l rep'n of W_E , i.e.

$$W_E \longrightarrow \hat{G}(\mathbb{L})$$

$$\forall \text{ alg rep'n } \gamma_V: \hat{G}(\mathbb{L}) \longrightarrow GL(V)$$

$$\rightsquigarrow \gamma_{V, *}(L): W_E \longrightarrow \hat{G}(\mathbb{L}) \longrightarrow GL(V)$$

Def $\mathcal{F} \in D_{\text{is}}(\text{Bun}_n, \overline{\mathbb{Q}_\ell})$ is called a Hecke eigensheaf with

$$\text{eigenvalue } \mathbb{L} \text{ if } \forall I, (V_i)_{i \in I} \in \text{Rep}(\hat{G})^I$$

$$\exists \eta_{(V_i)_{i \in I}}: T_{(V_i)_{i \in I}}^I(\mathcal{F}) \xrightarrow{\sim} \mathcal{F} \boxtimes \left(\prod_{i \in I} \gamma_{V_i, *}(L) \right)$$

natural in I , compatible with composition & exterior prod.

Conj. (Fargues)

\forall in $\overline{\mathbb{Q}_\ell}$ -rep'n \mathbb{L} of W_E of dim n , $\exists \text{Aut}_{\mathbb{L}} \in D_{\text{lis}}(\text{Bun}_G, \overline{\mathbb{Q}_\ell})$ s.t.

(1) $\text{Aut}_{\mathbb{L}}$ is a Hecke eigensheaf with eigenvalue \mathbb{L}

$$(2) \text{Aut}_{\mathbb{L}} \simeq \bigoplus_{b \in B(\mathbb{C})_{\text{basic}}} j_{b,1}(\mathcal{F}_{LL_b(\mathbb{L})})$$

$\mathcal{F}_{LL_b(\mathbb{L})} \in D_{\text{lis}}(\text{Bun}_G^b, \overline{\mathbb{Q}_\ell}) \simeq D(\text{Rep}_{\overline{\mathbb{Q}_\ell}}^\infty(G_b(E)))$ corresponding to $LL_b(\mathbb{L})$ via JL cor.
 $\left[\frac{*}{G_b(E)} \right]$

Main thm of [AL] Farques' conj. is true!

Main tool: Averaging functor

$$A_{V,\mathbb{L}}: D_{\text{lis}}(\text{Bun}_G, \overline{\mathbb{Q}_\ell}) \longrightarrow D_{\text{lis}}(\text{Bun}_M, \overline{\mathbb{Q}_\ell})$$

Constant term

$P \subset G = GL_n$ standard parabolic, $M \subset P \rightarrow M$

$$\text{Bun}_n \xleftarrow{p} \text{Bun}_P \xrightarrow{q} \text{Bun}_M$$

Lem (1) p is representable in loc. spatial diamonds

compactifiable & loc of dim $\text{trg}(p) < \infty$

(2) q is coh. sm.

Pf (1) $S \in \text{Perf}_k$, $E \in \text{Bun}_n(S)$

$$\text{Bun}_P \times_{\text{Bun}_n} S = \{ \text{red. of } E \text{ to } P \} = \mathcal{M}_Z$$

$$Z := E/P \longrightarrow X_{S,E}$$

[FS IV.4.2] \mathcal{M}_Z is rep in loc. spatial diamond

(2) WLOG $M = GL_{n_1} \times GL_{n_2}$

$$\text{Given } F \in \text{Bun}_M(S), F = E_1 \times E_2$$

$$\text{Bun}_P \times_{\text{Bun}_M} S = E_{X_1}(E_2, E_1): \text{Perf}_S \longrightarrow \text{Gpd}$$

$$T \longmapsto \{ 0 \rightarrow E_{1,T} \rightarrow E \rightarrow E_{2,T} \rightarrow 0 \}$$

WLOG S affinoïd, consider surjection

$$\mathcal{O}_{X_S}(-d)^N \twoheadrightarrow \mathcal{E}_2^\vee$$

$$\hookrightarrow 0 \rightarrow \mathcal{E}_2 \rightarrow \mathcal{O}_{X_S}(d)^N \rightarrow \mathcal{G} \rightarrow 0$$

Enlarging d , s.t. $\mathcal{E}_1^\vee \otimes \mathcal{O}_{X_S}(d)^N, \mathcal{E}_1^\vee \otimes \mathcal{G}$ have positive slopes.

$$X_1 := \text{Hom}(\mathcal{E}_1, \mathcal{O}_{X_S}(d)^N), X_2 = \text{Hom}(\mathcal{E}_1, \mathcal{G})$$

$$\text{Ext}^1(\mathcal{E}_1, \mathcal{E}_2) = X_2 / X_1 \quad \square$$

$\Lambda = \text{torsion } \mathbb{Z}_l[\sqrt{q}]\text{-alg}$

$$\text{Def } CT_{P,!} : D_{\text{ét}}(\text{Bim}_n, \Lambda) \longrightarrow D_{\text{ét}}(\text{Bim}_M, \Lambda)$$

$$\mathcal{F} \longmapsto q_! p^* \mathcal{F}$$

$$\text{Eis}_{P,*} : D_{\text{ét}}(\text{Bim}_M, \Lambda) \longrightarrow D_{\text{ét}}(\text{Bim}_n, \Lambda)$$

$$\mathcal{G} \longmapsto p_* q^! \mathcal{G}$$

$\hookrightarrow (CT_{P,!}, \text{Eis}_{P,*})$ adjoint pair

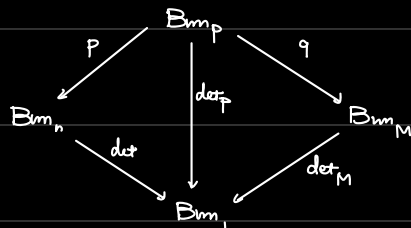
Def An $\mathcal{G} \in D_{\text{ét}}(\text{Bim}_n, \Lambda)$ is cuspidal if

$\forall P \subset G$ standard parabolic,

$$CT_{P,!}(\mathcal{G}) = 0$$

$\hookrightarrow D_{\text{ét}, \text{cusp}}(\text{Bim}_n, \Lambda)$

Twisting



lem $\forall \mathcal{G} \in D_{\text{ét}}(\text{Bim}_1), \mathcal{F} \in D_{\text{ét}}(\text{Bim}_n)$

$$CT_{P,!}(\mathcal{F} \otimes_{\Lambda}^{\mathbb{L}} \det^* \mathcal{G}) \xrightarrow{\sim} CT_{P,!}(\mathcal{F}) \otimes_{\Lambda}^{\mathbb{L}} \det_M^* \mathcal{G} \quad \square$$

lem Any $\mathcal{F} \in D_{\text{ét}, \text{cusp}}(\text{Bim}_n)$ is supported on Bim_n^{ss}

Pf $b \in B(G)$ non s.s., M standard Levi attached to b

$$b \in B(M)$$

$$B_{\mathfrak{p}}^b := B_{\mathfrak{p}} \times_{B_{\mathfrak{p},M}} B_M^b$$

$$\text{Then } D_{\mathfrak{e}_1}(B_{\mathfrak{p}}^b) \simeq D_{\mathfrak{e}_1}(B_M^b)$$

(Since fiber of $B_{\mathfrak{p}}^b \rightarrow B_M^b$ is a positive BC space)

$$\Rightarrow j_{b,*}(D_{\mathfrak{e}_1}([*]_{G_b(E)}], \wedge) \subset \text{Im}(Eis_{\mathfrak{p},*})$$

$$F \in D_{\mathfrak{e}_1, \text{cusp}}(B_{\mathfrak{p},n}), \quad g \in D_{\mathfrak{e}_1}([*]_{G_b(E)}])$$

$$j_{b,*}g = Eis_{\mathfrak{p},*}(g')$$

$$\text{Hom}(j_b^*F, g) \simeq \text{Hom}(F, j_{b,*}g) \simeq \text{Hom}(F, Eis_{\mathfrak{p},*}(g')) \simeq \text{Hom}(CT_{\mathfrak{p},!}F, g') = 0$$

Def $b \in B(G)$ basic, irr. $\pi \in \text{Rep}_{\Lambda}^{\infty}(G_b(E))$

(1) π is called cuspidal if

$$J_P(\pi) = 0, \quad \forall \text{ proper parabolic } P \subset G_b$$

$$J_P: \text{Rep}_{\Lambda}^{\infty}(G_b(E)) \rightarrow \text{Rep}_{\Lambda}^{\infty}(M)$$

$$\Leftrightarrow \pi \text{ is not a subrep of } \text{Ind}_P^{G_b}(\tau), \quad \tau \in \text{Rep}_{\Lambda}^{\infty}(M)$$

π is called supercuspidal if

$$\pi \text{ is not a subquotient of } \text{Ind}_P^{G_b}(\tau), \quad \tau \in \text{Rep}_{\Lambda}^{\infty}(M)$$

(When $\text{char } \Lambda = 0$, $\text{cusp.} = \text{supercusp.}$)

(2) π is geometrically cuspidal if

$$j_{b,!}(F_{\pi}) \in D_{\mathfrak{e}_1, \text{cusp}}(B_{\mathfrak{p},n})$$

Prop (1) π is geo. cusp. $\Rightarrow \pi$ is cusp.

$$(2) \quad n=2, \quad G_b \simeq CL_2 \quad (\text{i.e. } \nu(b) \in 2\mathbb{Z})$$

$$\pi \text{ is geo. cusp.} \Leftrightarrow \text{supercusp.}$$

Proof (1) $P_b \subset G_b$ parabolic with Levi M_b

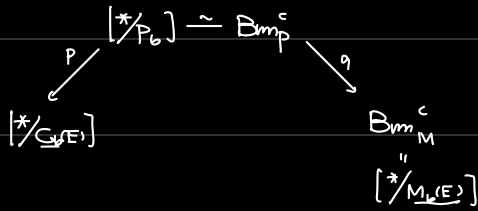
\exists parabolic $P \subset G$ with Levi M s.t. (G quasi-split)

$$\mathrm{Bim}_P \simeq \mathrm{Bim}_{P_b}, \quad \mathrm{Bim}_M \simeq \mathrm{Bim}_{M_b}$$

$c \in \mathrm{Bim}_M$ cov. to trivial M_b -torsor

$$[* / P_b] \simeq \mathrm{Bim}_P^c := \mathrm{Bim}_P^c := \mathrm{Bim}_P *_{\mathrm{Bim}_M} \mathrm{Bim}_M^c \longrightarrow \mathrm{Bim}_n$$

has image in Bim_n^b



$$\hookrightarrow \mathrm{CT}_{P,1}(\mathcal{F}_\pi)_c = J_{P_b}(\pi)$$

(2) Assume $n=2$, π geo. cusp. \Rightarrow supercuspidal

if not, by Vignéras, $q \equiv -1 \pmod{l}$

$$\pi = \pi_1 \otimes \chi \circ \det$$

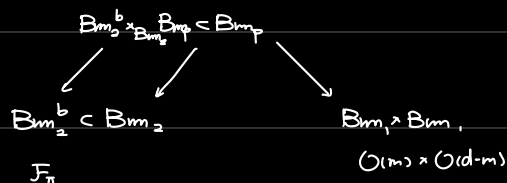
π_1 , unique ∞ -dim'l factor of

$$\mathrm{Ind}_B^{GL_2}(1 \boxtimes 1.1)$$

???

supercuspidal \Rightarrow geo. cusp.

$$\mathrm{CT}_{B,1}(\mathcal{F}_\pi) \cong 0$$



$$\Leftrightarrow \forall m, \mathcal{Z}_m := \mathrm{Hom}(\mathcal{E}_b, \mathcal{O}(m))$$

$$d = \kappa(b) \in 2\mathbb{Z}$$

\cup

$$\mathcal{Z}_m^0 = \mathrm{Hom}^{\mathrm{smj}}(\mathcal{E}_b, \mathcal{O}(m))$$

$$\mathrm{RT}_\eta(GL_2(E), \mathrm{RT}_c(\tilde{\mathcal{Z}}_m^0, \wedge) \otimes \pi) = 0$$

where $\tilde{Z}_m^0 \rightarrow Z_m^0$
 $(0 \rightarrow \mathcal{O}(d-m) \rightarrow \mathcal{E}_b \rightarrow \mathcal{O}(cm) \rightarrow 0)$

Averaging functor

$$\begin{array}{ccc} \text{Div}' & \xleftarrow{\alpha} \text{Mod}'_n: S \longrightarrow & \{ (x, \mathcal{E} \hookrightarrow \mathcal{E}') \mid x \in X_S \} \\ \swarrow \tilde{h} & & \searrow \tilde{h} \\ \text{Bm}_n & & \text{Bm}_n \end{array} \quad \cdot \{ 0 \rightarrow \mathcal{E} \rightarrow \mathcal{E}' \rightarrow i_* \mathcal{C}_x \rightarrow 0 \}$$

Def $\mathbb{L} \in D_{\mathcal{E}_1}(\text{Div}', \Lambda)$

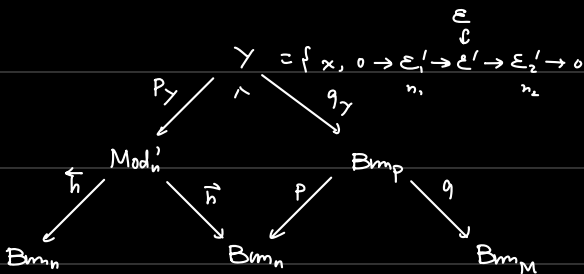
$$\begin{aligned} \text{Av}_{\mathbb{L}, n}: D_{\mathcal{E}_1}(\text{Bm}_n) &\longrightarrow D_{\mathcal{E}_1}(\text{Bm}_n) \\ \mathcal{F} &\longmapsto \tilde{h}_! (\tilde{h}^* \mathcal{F}) \otimes_{\Lambda}^{\mathbb{L}} \alpha^* \mathbb{L}(\frac{n-1}{2}[n-1]) \end{aligned}$$

Prop $P = \begin{pmatrix} n_1 & * \\ & n_2 \end{pmatrix} \subset G = GL_n$

$M = GL_{n_1} \times GL_{n_2}$

$\exists (\text{Av}_{\mathbb{L}(\frac{n_1}{2}[n_1], n_1} \times \text{Id}_{\text{Bm}_{n_2}}) \circ \text{CT}_{P,1} \longrightarrow \text{CT}_{P,1} \circ \text{Av}_{\mathbb{L}, n} \longrightarrow (\text{Id}_{\text{Bm}_{n_1}} \times \text{Av}_{\mathbb{L}(\frac{n_2}{2}[n_2], n_2)}) \circ \text{CT}_{P,1} \xrightarrow{+1}$ dist. triangle.

Proof



$U \hookrightarrow \gamma, \quad Z = \gamma/U$

$\{ (x, 0 \rightarrow \mathcal{E}'_1 \rightarrow \mathcal{E}'_2 \rightarrow 0) \mid \mathcal{E} \rightarrow \mathcal{E}'_1 \rightarrow \mathcal{C}'_2 \text{ surjective} \}$

$\text{CT}_{P,1} \text{Av}_{\mathbb{L}, n}(\mathcal{F}) = (q \circ q_\gamma)_! ((p_\gamma^* \circ \tilde{h}^* \mathcal{F}) \otimes \mathbb{L}_\gamma)$

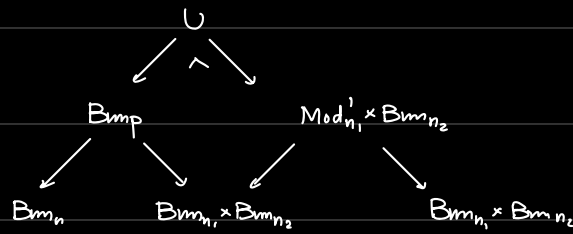
$\begin{array}{c} j_! j^* \mathbb{L}_\gamma \rightarrow \mathbb{L}_\gamma \rightarrow i_* i^* \mathbb{L}_\gamma \xrightarrow{+1} \\ \uparrow \text{ii} \\ \mathbb{L}_U \end{array}$

Claim $(q \circ q_\gamma)_! ((p_\gamma^* \circ \tilde{h}^* \mathcal{F}) \otimes \mathbb{L}_U) = (\text{Av} \times \text{Id}_{\text{Bm}_{n_2}}) \circ \text{CT}_{P,1}$

Key $\begin{array}{c} 0 \rightarrow \mathcal{E}'_1 \rightarrow \mathcal{E}'_2 \rightarrow \mathcal{E}'_3 \rightarrow 0 \\ \uparrow \quad \parallel \\ 0 \rightarrow \mathcal{E}_1 \rightarrow \mathcal{E} \rightarrow \mathcal{E}_2 \rightarrow 0 \end{array}$

$\hookrightarrow (x, \mathcal{E}_1 \hookrightarrow \mathcal{E}'_1) \in \text{Mod}'_n$

The correspondence by \mathbb{L}_U is identified with



Similar over Z

Prop $\chi: W_E \rightarrow \Lambda^* \xrightarrow{\sim} \mathbb{L}_\chi \in D_{\text{ét}}(\text{Div}^1, \Lambda)$

Let $G_j \in D_{\text{ét}}(Bm_n, \Lambda)$ be a Hecke eigenstate with eigenvalue \mathbb{L}_χ . Then $\forall F \in D_{\text{ét}}(Bm_n)$

$$Av_{\mathbb{L}, n}(F \otimes_{\Lambda} \det^* G_j) \simeq Av_{\mathbb{L} \otimes \mathbb{L}_\chi^v}(F) \otimes \det^* G_j$$

Thm $\chi: W_E \rightarrow \Lambda^* \xrightarrow{\sim} \mathbb{L}_\chi$

Then $\forall F \in D_{\text{ét}, \text{cusp}}(Bm_n, \Lambda)$, $Av_{\mathbb{L}_\chi}(F) = 0$

Rmk In general, Fontaine's conj. suggests if \mathbb{L} is a Λ -l.s. of $\text{rk} \leq n$.

then $Av_{\mathbb{L}, n}(F) = 0 \quad \forall F \in D_{\text{ét}, \text{cusp}}(Bm_n)$

Pf WMA $\mathbb{L} = \Lambda$ trivial by previous prop.

Prop 1 $\Rightarrow F, Av(F) \in D_{\text{ét}, \text{cusp}}$

$\Rightarrow \text{supp}(F), \text{supp}(Av F) \subset Bm_n^{\text{ss}}$

suffices to show $\forall b, c, \kappa(b) = \kappa(c) + 1$

$\forall F \in D_{\text{ét}, \text{cusp}}$ supported on Bm_n^b

$j_c^* Av(F) = 0$

$\forall r=0, \dots, n, d \in \mathcal{B}(GL_r)$

$Z_{r,d} = \{ \mathcal{E} \rightarrow \mathcal{F} \rightarrow \mathcal{E}' \mid \mathcal{E} \simeq \mathcal{E}_b, \mathcal{E}' \simeq \mathcal{E}_c, \mathcal{F} \simeq \mathcal{E}_d \}$

$j_{r,d}: Z_{r,d} \rightarrow Z \quad |Z| = \bigsqcup_{r,d} |Z_{r,d}|$

$\begin{matrix} & & \searrow q_1 & & \searrow q_2 \\ & & Bm_n & & Bm_n \end{matrix}$

$$A_{V,r,d}: D_{\mathbb{Z}_d}(B_{m,n}) \longrightarrow D_{\mathbb{Z}_d}(B_{m,n})$$

$$g \longmapsto q_2! q_1^* g \sigma_{\Lambda}^{\pm} j_{c,d}! \wedge$$

$$\bar{A}_V: g \longmapsto q_2! q_1^* g$$

$$A_V = A_{V,n,b}, \quad \mathbb{Z}_{n,d} = \emptyset \text{ if } d \neq b$$

Claim Any of $A_{V,r,d}(F)$, $\bar{A}_V(F)$ vanishes if $r < n$

$$\text{For } \bar{A}_V: \mathbb{Z}^*_{B_{m,n}} k = [G_b(E) \setminus \text{Hom}(\mathcal{E}_b, \mathcal{E}_c)]$$

↑
positive BC space

$$j_c^* \bar{A}_V(F) = R\Gamma_c(\mathbb{Z}^*_{B_{m,n}} k, q_1^* F)$$

$$= R\Gamma(G_b(E), F_b) \neq 0$$

• $\chi(b) \in 2\mathbb{Z} \Rightarrow R\Gamma(G_b(E), F_b)$ vanishes as F_b supersingular

• $\chi(b) \in \mathbb{Z} \setminus 2\mathbb{Z} \Rightarrow = 0$ as $G_b(E)$ cpt mod center

$$A_{V,r,d}: \mathbb{Z}_{r,d} \times_{B_{m,n}} k = [G_b(E) \setminus \text{Hom}^{smj}_c(\mathcal{E}_b, \mathcal{E}_d) \times^{g_d} \text{Hom}^{inj}_c(\mathcal{E}_d, \mathcal{E}_c)]$$

$$\downarrow \text{ fibration } \\ [g_d \setminus \text{Hom}^{inj}_c(\mathcal{E}_d, \mathcal{E}_c)]$$

$$R\Gamma_c(\mathbb{Z}_{r,d} \times_{B_{m,n}} k, q_1^* F) \neq 0$$

reduce to show $R\Gamma_c([G_b(E) \setminus \text{Hom}^{smj}_c(\mathcal{E}_b, \mathcal{E}_d)], q_1^* F) \neq 0$

$$R\Gamma_c^p \text{CT}_{P_{r,n-r},!}(F)_{\mathcal{E}_d}$$

$$B_{m,r} \times B_{m,n-r} \xrightarrow{f} B_{m,r}$$

$\downarrow p_d$

□