Fargues' Conjecture

Notation

$L = \overline{\mathbb{Q}}_\ell$, $G = GL_n/_E$, $\widehat{G} = GL_n/_L$

$E/\mathbb{Q}_p$ finite ext'n , W_E = Weil gp

For $\mathbb{L}$ an irr cts W_E-rep'n, $b \in B(G)_{basic}$, denote $LL_b(\mathbb{L}) \in Rep^\infty(G_b(E))$ to be its local Langlands correspondence

$$X_{\widehat{G}} := [Z^1(W_E, \widehat{G})/\widehat{G}] \xrightarrow{f} [*/\widehat{G}]$$

$Bun_n = \coprod_{b \in B(G)} Bun_n^b$, $Bun_n^{ss} = \coprod_{b \in B(G)_{basic}} Bun_n^b$, $B(G)_{basic} \xrightarrow[\sim]{\kappa} \mathbb{Z}$, $j_b : Bun_n^b \hookrightarrow Bun_n$, $D_{lis}(Bun_n^b, L) \simeq D(Rep^\infty(G_b(E)))$

Main thm (Fargues' conjecture)

Assume $\mathbb{L}$ is an irreducible continuous W_E-rep'n of rk n over L

Then there exists a non-zero Hecke eigensheaf $Aut_{\mathbb{L}} \in D_{lis}(Bun_G, L)$ with eigenvalue $\mathbb{L}$

($i.e.$ $V \in Rep(G^\vee)^I \rightsquigarrow T_V(Aut_{\mathbb{L}}) \xrightarrow{\sim} r_V(\mathbb{L}) \boxtimes Aut_{\mathbb{L}}$)

s.t. • $Aut_{\mathbb{L}}$ is supported on Bun_n^{ss}

• For each $b \in B(G)$ basic , $j_b^* Aut_{\mathbb{L}} \simeq \mathcal{F}_{LL_b(\mathbb{L})}$

Construction of $Aut_{\mathbb{L}}$

Take $\psi : E \longrightarrow L^\times$ non-trivial character

$\rightsquigarrow \psi : U(E) \longrightarrow (U/_{[U,U]})(E) \xrightarrow{sum} E \xrightarrow{\psi} L^\times$

$\rightsquigarrow$ Whittaker sheaf $W_\psi := j_{1,!}(\text{c-Ind}_{U(E)}^{G(E)} \psi)$

Recall one has spectral action $Perf(X_{G^\vee}) \overset{*}{\curvearrowright} D_{lis}(Bun_G, L)$

View $\mathbb{L} \in X_{G^\vee}(L) \rightsquigarrow i_{\mathbb{L}} : Spec\, L \longrightarrow X_{G^\vee}$

$\mathbb{G}_m$-weight decomposition

$k(\mathbb{L})_{reg} := (i_{\mathbb{L}})_* L = \bigoplus_{n \in \mathbb{Z}} k(\mathbb{L})_n$, define $Aut_{\mathbb{L}} := k(\mathbb{L})_{reg} * W_\psi$

Check $Aut_{\mathbb{L}}$ is eigen

$f : X_{G^\vee} \longrightarrow [*/G^\vee]$.

$T_V(k(\mathbb{L})_{reg} * W_\psi) = f^*V * k(\mathbb{L})_{reg} * W_\psi = r_{V,*}(\mathbb{L}) \otimes k(\mathbb{L})_{reg} * W_\psi$

How to compute $Aut_{\mathbb{L}}$?

Or equivalently, how to compute $k(\mathbb{L})_{reg} *$?

Recall $Av_{\mathbb{L}^\vee} : D_{lis}(Bun_G, L) \longrightarrow D_{lis}(Bun_G, L)$

$$\mathcal{F} \longmapsto (T_V(\mathcal{F}) \underset{L}{\otimes} \mathbb{L}^\vee)^{W_E}$$

$\mathcal{A}v_{\mathbb{L}^\vee} := (f^*V_{std} \otimes \mathbb{L}^\vee)^{W_E} \in Perf(X_{\hat{G}^\vee})$

Then $Av_{\mathbb{L}^\vee} = \mathcal{A}v_{\mathbb{L}^\vee} *$

Lemma 7.6 $\mathcal{A}v_{\mathbb{L}^\vee} = k(\mathbb{L})_![-1] \oplus k(\mathbb{L}(\chi_{cycl}))_![-2]$ Idea Check on (derived) stalks.

Proof For $\mathbb{L}' \in X_{\hat{G}^\vee}(L)$,

$$i_{\mathbb{L}'}^* \mathcal{A}v_{\mathbb{L}^\vee} = (\underbrace{\mathbb{L}' \otimes \mathbb{L}^\vee}_{\mathbb{G}_m \text{ wt } 1})^{W_E} \simeq \begin{cases} L \oplus L[-1] & \mathbb{L}' \simeq \mathbb{L} \\ L[-1] \oplus L[-2] & \mathbb{L}' \simeq \mathbb{L}(\chi_{cycl}) \\ 0 & \text{else} \end{cases} \qquad (*)$$

Recall [FS X.2] $C_{\mathbb{L}} :=$ connected component of $X_{\hat{G}^\vee}$ containing $\mathbb{L}$

$$C_{\mathbb{L}}(L) \overset{\sim}{\longleftrightarrow} \{\text{unramified twists of } \mathbb{L}\}$$
$$\mathbb{L}' \simeq \mathbb{L} \otimes \chi \quad \text{for } \chi : W_E \twoheadrightarrow \mathbb{Z} \to L^\times \simeq Z(GL_1)(L)$$

trivial action

$$C_{\mathbb{L}} \approx [\operatorname{Spec} L[t,t^{-1}] / \mathbb{G}_m] \qquad t=1 \longleftrightarrow \mathbb{L}$$

compute derived stalks?

Now, $(*) \Rightarrow \mathcal{A}v_{\mathbb{L}^\vee}|_{C_{\mathbb{L}}} \simeq (L[t,t^{-1}]/(t-1)^n)_![-1]$ for some $n \geq 1$

$\mathbb{G}_m$-weight

$$\mathcal{A}v_{\mathbb{L}^\vee} / {}^{\mathbb{L}}(t-1)^2 \simeq \left(\begin{pmatrix} \mathbb{L} & ** 0 \\ & \mathbb{L} \end{pmatrix} \otimes \mathbb{L}^\vee\right)^{W_E} \simeq L \oplus L[-1] \Rightarrow n = 1$$

$\Rightarrow \mathcal{A}v_{\mathbb{L}^\vee}|_{C_{\mathbb{L}}} \simeq k(\mathbb{L})_![-1]$ Similar for $\mathcal{A}v_{\mathbb{L}^\vee}|_{C_{\mathbb{L}(\chi_{cycl})}} \simeq k(\mathbb{L}(\chi_{cycl}))_![-2]$ $\square$

Key $k(\mathbb{L})_! * \mathcal{G}$ is a direct summand of $Av_{\mathbb{L}^\vee}[1]$

Lemma 7.7 The following are true for $i \in \mathbb{Z}$

degree

(1) $\mathcal{G} \in D_{lis}(Bun_n, L)$ is supported on $Bun_n^{\kappa=d}$

$\Rightarrow$ $k(\mathbb{L})_i * \mathcal{G}$ is supported on $Bun_n^{\kappa=d+i}$

(2) $\mathcal{G} \in D_{lis}(\mathrm{Bun}_n, L) \Rightarrow$ • $k(\mathbb{L})_i * \mathcal{G}$ is supported on Bun_n^{ss}

• $j_b^*(k(\mathbb{L})_i * \mathcal{G})$ are supercuspidal for $b \in B(G)_{basic}$

(3) $k(\mathbb{L})_i * \mathcal{G} = 0$ if either • $\mathcal{G}$ is supported on $\mathrm{Bun}_n \setminus \mathrm{Bun}_n^{ss}$

or • $\mathcal{G} = j_{b,!}\mathcal{F}_\pi$ for some parabolically induced $\pi \in D(\mathrm{Rep}^\infty(G_b(E)))$

<u>Proof</u> (2), (3) doesn't rely on $\mathrm{Av}_{\mathbb{L}^\vee}[1]$

Take $e \in \mathcal{O}(X_{\hat{G}}) = Z^{spec}(G, L)$ to be 1 on $C_{\mathbb{L}}$, 0 elsewhere

<u>Claim</u> e acts by 0 on $\mathcal{G} \in D_{lis}(\mathrm{Bun}_n, L)$ s.t.

either ① $\mathcal{G}$ is supported on $\mathrm{Bun}_n \setminus \mathrm{Bun}_n^{ss}$

② $\mathcal{G} = j_{b,!}\mathrm{Ind}_{P(E)}^{G_b(E)}\sigma$ for $\sigma \in D(\mathrm{Rep}^\infty(P))$ & parabolic $P \subset G_b$, $b \in B(G)_{basic}$

① Recall $b \in B(G) \rightsquigarrow \hat{G}_b \rtimes Q \longrightarrow \hat{G} \rtimes Q$ Levi

$\rightsquigarrow Z^{spec}(G, \Lambda) \longrightarrow Z^{spec}(G_b, \Lambda)$

<u>[FS Thm IX.7.2]</u> Compatibility with G_b

$$\begin{tikzcd} Z^{spec}(G,\Lambda) \arrow[r, "\Psi_G^b"] \arrow[d] \arrow[dr, phantom, "\circlearrowleft"] & Z(D(G_b(E),\Lambda)) \\ Z^{spec}(G_b,\Lambda) \arrow[ur, "\Psi_{G_b}"'] & \end{tikzcd} \quad (*)$$

$e \mapsto 0$

commutes

② Recall $P \subset G$, $P \twoheadrightarrow M$ $\rightsquigarrow {}^L M \longrightarrow {}^L G$ Levi

$\rightsquigarrow Z^{spec}(G, \Lambda) \longrightarrow Z^{spec}(M, \Lambda)$

<u>[FS Thm IX.7.3]</u> For $\sigma \in D(M(E), \Lambda)$

$$\begin{tikzcd} Z^{spec}(G,\Lambda) \arrow[r] \arrow[d] \arrow[dr, phantom, "\circlearrowright"] & \mathrm{End}(\mathrm{Ind}_{P(E)}^{G(E)}\sigma) \\ Z^{spec}(M,\Lambda) \arrow[r] & \mathrm{End}(\sigma) \arrow[u] \end{tikzcd}$$

$e \mapsto 0$

commutes □

<u>Proof of (3)</u> e acts by $e|_{\mathbb{L}} = 1$ on $k(\mathbb{L})_i$ $\Rightarrow k(\mathbb{L})_i * \mathcal{G} = 0$

0 on $\mathcal{G}$

<u>Proof of (2)</u> Denote $\mathcal{F} := k(\mathbb{L})_{reg} * \mathcal{G}$, more generally

Lemma 7.1 $\mathcal{F}$ is ULA with Hecke eigenvalue $\mathbb{L}$ $\Rightarrow$ same conclusion

$\mathcal{F}$ has eigenvalue $\mathbb{L}$ $\Rightarrow$ e acts by $e|_{\mathbb{L}} = 1$ on $\mathcal{F}$ (*)

$$j_{b,!}j_b^*\mathcal{F} \longrightarrow \overset{e=1}{\mathcal{F}} \xrightarrow{0} \overset{e=0}{i_{b,!}i_b^*\mathcal{F}} \longrightarrow \qquad \Rightarrow \; i_b^*\mathcal{F} = 0 \quad \text{for } b \in B(G)_{basic}$$

(*) Explanation For excursion datum $\gamma_I \in W_E^I$, $V_I \in Rep(\hat{G}^I)$, $1 \xrightarrow{\alpha} V^{\otimes I} \xrightarrow{\beta} 1$

$$\text{Function}\,(\rho: W_E \to \hat{G}) \longmapsto (1 \xrightarrow{\alpha} V^{\otimes I} \xrightarrow{\rho(\gamma_I)} V^{\otimes I} \longrightarrow 1)$$

acts on $\mathcal{F}$ via

$$\begin{tikzcd}
\mathcal{F} \arrow[r, "\alpha"] \arrow[dr, "\alpha"'] & T_{V^I}(\mathcal{F}) \arrow[r, "\gamma_I"] \arrow[d, phantom, "\cong"] & T_{V^I}(\mathcal{F}) \arrow[r, "\beta"] \arrow[d, phantom, "\cong"] & \mathcal{F} \\
& \gamma_{V_I,*}(\mathbb{L}^I) \otimes \mathcal{F} \arrow[r, "\rho_{\mathbb{L}}(\gamma_I)"'] & \gamma_{V_I,*}(\mathbb{L}^I) \otimes \mathcal{F} \arrow[ur, "\beta"'] &
\end{tikzcd}$$

Proof of (1) $k(\mathbb{L})_{!*}\mathcal{G}$ is a direct summand of $(Av_{\mathbb{L}^\vee}[1])^{(i)}$ □

Computation of $k(\mathbb{L})_{reg} * W_\psi = Aut_{\mathbb{L}}$

Thm 7.8 Suppose $b \in B(G)$, $\kappa(b) = 1$

$$j_b^* Aut_{\mathbb{L}} \simeq \mathcal{F}_{LL_b(\mathbb{L})}$$

Proof WTS $k(\mathbb{L})_{!*}W_\psi = j_{b,!}\mathcal{F}_{LL_b(\mathbb{L})}$

Claim $Av_{\mathbb{L}^\vee}(W_\psi)[1] = \mathcal{F}_{LL_b(\mathbb{L})} \oplus \mathcal{F}_{LL_b(\mathbb{L}(\chi_{cycl}))}[-1]$ (*)

Assuming the claim. RHS $= k(\mathbb{L})_{!*}W_\psi \oplus k(\mathbb{L}(\chi_{cycl}))_{!*}W_\psi[-1]$

To prove $k(\mathbb{L})_{!*}W_\psi = \mathcal{F}_{LL_b(\mathbb{L})}$, suppose not.

(*) $\Rightarrow$ $k(\mathbb{L})_{!*}W_\psi \in D^{\geq 0}(Rep^\infty(G_b(E)))$

$k(\mathbb{L}(\chi_{cycl}))_{!*}W_\psi \in D^{\leq 0}(Rep^\infty(G_b(E)))$ similarly. $k(\mathbb{L})_{!*}W_\psi \in D^{\leq 0}(Rep^\infty(G_b(E)))$

$\Rightarrow$ $k(\mathbb{L})_{!*}W_\psi \in D(Rep^\infty(G_b(E)))^\heartsuit$ hence $= \mathcal{F}_{LL_b(\mathbb{L})}$

Proof of claim

Input Thm 7.8 ([FS Thm IX.7.4])

$$j_b^* T_{V_{std}}(j_{1,!}\mathcal{F}_{LL_1(\mathbb{L})}) \simeq \mathbb{L} \otimes_{\bar{\mathbb{Q}}_\ell} \mathcal{F}_{LL_b(\mathbb{L})}$$

$\Rightarrow$ For $\pi \in Rep^\infty(GL_n(E))$ irr. supercuspidal, suppose $\pi = LL_1(\mathbb{L}')$

$$j_b^* Av_{\mathbb{L}^\vee}(j_{1,!}\mathcal{F}_\pi) = (\mathbb{L}^\vee \otimes \mathbb{L}' \otimes \mathcal{F}_{LL_b(\mathbb{L}')})^{W_E} = \begin{cases} \mathcal{F}_{LL_b(\mathbb{L}')} \oplus \mathcal{F}_{LL_b(\mathbb{L}')}[-1] & \text{if } \mathbb{L}' = \mathbb{L} \\ \mathcal{F}_{LL_b(\mathbb{L}')}[-1] \oplus \mathcal{F}_{LL_b(\mathbb{L}')}[-2] & \text{if } \mathbb{L}' = \mathbb{L}(\chi_{cycl}) \\ 0 & \text{else} \end{cases} \qquad (*)$$

- $Av_{\mathbb{L}^\vee}$ kills blocks of $Rep^\infty(G(E))$ except that containing $LL_1(\mathbb{L})$, $LL_1(\mathbb{L}(\chi_{cycl}))$
  and has image lies in blocks of $D(Rep^\infty(G_b(E)))$ containing $LL_b(\mathbb{L})$, $LL_b(\mathbb{L}(\chi_{cycl}))$

Fix $\mathcal{B} \subset Rep^\infty(G(E))$ supercuspidal block containing $LL_1(\mathbb{L})$

$\mathcal{C} \subset Rep^\infty(G_b(E))$ supercuspidal block containing $LL_b(\mathbb{L})$

For $V \in Rep^\infty(G(E))$. denote $V_{\mathcal{B}}$ the summand of V in $\mathcal{B}$

Fact $\mathcal{B} \simeq Mod_{\mathfrak{Z}}$ for $\mathfrak{Z} := End_{\mathcal{B}}(W_{\psi,\mathcal{B}}) \simeq L[t,t^{-1}]$ WLOG $LL_1(\mathbb{L}) \leftrightarrow \mathfrak{Z}/(t-1)$

$V \longmapsto Hom_{\mathcal{B}}(W_{\psi,\mathcal{B}}, V)$

$\mathcal{C} \simeq Mod_R$ for $R \simeq L[s,s^{-1}]$ $LL_b(\mathbb{L}) \rightsquigarrow R/(s-1)$

$\rightsquigarrow F_{\mathcal{B},\mathcal{C}} : \mathcal{B} \subset D_{lis}(Bun_n, \bar{\mathbb{Q}}_\ell) \xrightarrow{Av_{\mathbb{L}^\vee}[1]} D_{lis}(Bun_n, \bar{\mathbb{Q}}_\ell) \xrightarrow{P_{\mathcal{C}} \circ j_b^*} D(\mathcal{C})$, suffices $F_{\mathcal{B},\mathcal{C}}(W_{\psi,\mathcal{B}}) \simeq LL_b(\mathbb{L})$

i.e. $M \simeq R/(s-1)$ as R-module

$F_{\mathcal{B},\mathcal{C}}$ commutes with colimit $\Rightarrow$ $F_{\mathcal{B},\mathcal{C}} \simeq M \otimes^{\mathbb{L}}_{\mathfrak{Z}} (-)$ for some $(R,\mathfrak{Z})$-bimodule M

$F_{\mathcal{B},\mathcal{C}}$ commutes with limit + preserves cpt obj ? $\Rightarrow$ M is perfect $\mathfrak{Z}$-module

(preserves cpt obj $\Rightarrow$ M is perfect R-module)

Then $(*) \Rightarrow M/^{\mathbb{L}}(t-a) \simeq \begin{cases} R/(s-1) \oplus R/(s-1)[1] & a = 1 \\ 0 & a \neq 1 \end{cases}$ as R-module

$\Rightarrow M \approx \mathfrak{Z}/(t-1)^n$ for $n \geq 1$?

Since $\mathcal{H}^0(M/^{\mathbb{L}}(t-1)^2) \simeq L$, we know $M \simeq \mathfrak{Z}/(t-1)$ $\square$

<u>Explanation</u> Let $\pi = \begin{pmatrix} LL_1(\mathbb{L}) & * \neq 0 \\ & LL_1(\mathbb{L}) \end{pmatrix}$

$$j_b^*(T_{V_{std}}(j_{1,!}\mathcal{F}_\pi)) = \begin{pmatrix} \mathbb{L}\otimes LL_b(\mathbb{L}) & * \\ & \mathbb{L}\otimes LL_b(\mathbb{L}) \end{pmatrix}$$

<u>Claim</u> $* \neq 0$

Assuming this, $\mathcal{H}^0(Av_{\mathbb{L}^\vee}(\mathcal{F}_\pi)) \simeq LL_b(\mathbb{L})$, so $M \not\simeq \mathcal{Z}/(t-1)^n$ for $n \geq 2$

<u>Proof of claim</u> Otherwise, $T_{V_{std}}(j_{1,!}\mathcal{F}_\pi) \simeq (\mathbb{L}\otimes j_{b,!}\mathcal{F}_{LL_b(\mathbb{L})})^{\oplus 2}$

$$\Rightarrow T_{V_{std}^*}(T_{V_{std}}((j_{1,!}\mathcal{F}_\pi)) \simeq (\mathbb{L}^*\otimes\mathbb{L}\otimes j_{1,!}\mathcal{F}_{LL_1(\mathbb{L})})^{\oplus 2}$$

which contains $j_{1,!}\mathcal{F}_\pi$ as a direct summand × □

To determine $j_b^* \mathrm{Aut}_{\mathbb{L}}$ for $\kappa(b) \neq 1$

<u>Step 1</u> <u>Lemma 7.2</u> Suppose $\mathcal{F} \in D_{lis}(Bun_n, L)$ has Hecke eigenvalue $\mathbb{L}$, and for some $b \in B(G)_{basic}$, $\mathcal{F}_b := j_b^*\mathcal{F}$ is irreducible, then $\mathcal{F}_c$ is <u>irreducible</u> for all $c \in B(G)_{basic}$

a shift of irr. rep'n

<u>Proof</u> WLOG $\kappa(c) = \kappa(b) + 1$

$$T_{V_{std}^*}(\mathcal{F}_c) \simeq \mathbb{L}^*\otimes\mathcal{F}_b$$

Since $\mathbb{L}^*\otimes\mathcal{F}_b$ is irreducible as $W_E \times G_b(E)$-rep'n, we know

<u>$T_{V_{std}^*}$ kills all Jordan-Hölder factors of cohomology sheaves of $\mathcal{F}_c$ except one</u> ?

2 explanations

① [Han] $\Rightarrow T_{V_{std}^*}$ is t-exact on $D(Bun_{G}, L)^{\mathbb{L}}$

② L-parameters of JH factors of $\mathcal{F}_c$ are $\mathbb{L}$, which coincide with usual L-par by by [HKW] $\Rightarrow$ factors are isomorphic

But $\mathcal{F}_c$ is a direct summand of $T_{V_{std}}(T_{V_{std}^*}(\mathcal{F}_c))$

$\Rightarrow \mathcal{F}_c$ has only one JH factor □

<u>Step 2</u> <u>Claim</u> $\mathcal{F}_c \simeq LL_c(\mathbb{L})[k_c]$ for $k_c \in \mathbb{Z}$

<u>Proof</u> Assume $\mathcal{F}_b \simeq LL_b(\mathbb{L})[k_b]$

For $\kappa(c) = \kappa(b) + 1$

$$\mathbb{L}\otimes\mathcal{F}_c \simeq T_{V_{std}}(LL_b(\mathbb{L})[k_b])$$

$$\left.\begin{aligned} &[HKW] \Rightarrow tr(\mathcal{F}_c) = (-1)^{k_b} tr(LL_c(\mathbb{L})) \in \mathrm{Dist}(G_c(E)_{ell}, L)^{G_c(E)} \\ &\mathcal{F}_c \text{ is irr supersingular} \end{aligned}\right\} \Rightarrow \mathcal{F}_c \simeq \mathcal{F}_{LL_c(\mathbb{L})}[k_c]$$

□

Step 3 [Ham] $\Rightarrow$ $T_{V_{std}}(LL_b(\mathbb{L})) \in \mathrm{Rep}^{\infty}(G_c(E))$

For $\kappa(b)=1$, $k_b = 0$

$\Rightarrow$ all $k_b = 0$ for $b \in G_b(E)$

Vanishing result

Ref [Hansen] On the supercuspidal cohomology of basic local Shimura varieties

Thm 1.1 (G,μ,b) basic local Shimura datum. ρ = supercuspidal rep'n of $G_b(\mathbb{Q}_p)$.

Suppose (1) $Sh(G,\mu,b)_K$ occur in basic uniformization at p of a global Shimura variety

(2) The L-parameter $\varphi_\rho : W_{\mathbb{Q}_p} \longrightarrow {}^LG(L)$ is supercuspidal

Then $H^i_c(G,\mu,b)[\rho] = 0$ for all $i \neq d = \dim Sh(G,\mu,b)_K$

Wher $H^i_c(G,\mu,b)[\rho] = H^i(R\Gamma_c(G,\mu,b)[\rho])$

$$R\Gamma_c(G,\mu,b)[\rho] = \operatorname*{colim}_K R\Gamma_c(Sh(G,\mu,b)_K, L) \otimes^{\mathbb{L}}_{\mathcal{H}(G_b(\mathbb{Q}_p))} \rho$$

$$\simeq \operatorname*{colim}_K R\mathrm{Hom}_{G_b(\mathbb{Q}_p)}(R\Gamma_c(Sh(G,\mu,b)_K, L), \rho^*)^*$$

$$= \operatorname*{colim}_K (R\mathrm{Hom}_{G_L(\mathbb{Q}_p)}(j_b^* T_{V_\mu} j_{1,!}\, c\text{-}\mathrm{Ind}_K^{G(\mathbb{Q}_p)} 1, \rho^*))^*$$

$$= \operatorname*{colim}_K ((j_1^* T_{V_\mu^*} j_{b,*} \mathcal{F}_{\rho^*})^K)^*$$

$$= (j_1^* T_{V_\mu^*} j_{b,*} \mathcal{F}_{\rho^*})^*$$

$$= j_1^* T_{V_\mu^*} j_{b,!} \mathcal{F}_\rho$$

In our case, take $G = \mathrm{Res}_{E/\mathbb{Q}_p} G_b$

Condition (1)

Prop 3.1 Fix $\overline{\mathbb{Q}}_p \simeq \mathbb{C}$

For a Shimura datum $(G, X) \rightsquigarrow \mu : \mathbb{G}_{m,\overline{\mathbb{Q}}_p} \to G_{\overline{\mathbb{Q}}_p}$, $b \in B(G,\mu)$

there is a canonical $\mathbb{Q}$-inner form G' of G s.t.

(1) $G'_{\mathbb{A}_f^p} \simeq G_{\mathbb{A}_f^p}$

(2) $G'_{\mathbb{Q}_p} \simeq G_b$

(3) $G_{\mathbb{R}}$ is compact modulo center

For open cpt $K \subset G(\mathbb{A}^f)$, $S(G,K)_K$ = rigid analytic SV$/\mathbb{C}_p$

$$S(G,X)_{K^p} := \varprojlim_{K_p} S(G,X)_{K^pK_p} \xrightarrow{\pi_{HT}} \mathcal{F}\ell_{G,\mu} \quad /\mathrm{Spd}\,\mathbb{C}_p$$

<u>Def 3.2</u> Say (G,X) satisfies basic uniformization at p if $\exists$ $G(\mathbb{A}_f)$-equivariant isom.

$$\varprojlim_{K^p} S(G,X)^b_{K^p} \simeq \left(\underline{G'(\mathbb{Q})}\backslash G'(\mathbb{A}_f) \times_{\mathrm{Spd}\,\mathbb{C}_p} \mathrm{Sh}(G,\mu,b)_\infty\right) / \underline{G_b(\mathbb{Q}_p)}$$

Where $\mathcal{F}\ell^b_{C,\mu} \simeq \mathrm{Sh}(G,\mu,b)_\infty / \underline{G_b(\mathbb{Q}_p)}$

$\mathrm{Sh}(G,\mu,b)_\infty : \mathrm{Perf}_{/C^b} \longrightarrow \mathrm{Sets}$

$S \longmapsto \{\mathcal{E}_b \dashrightarrow \mathcal{E}_1 \text{ meromorphic of type } \mu \text{ on } S^\#\}$

$\mathcal{F}\ell_{C,\mu} : \mathrm{Perf}_{/C^b} \longrightarrow \mathrm{Sets}$

$S \longmapsto \{(\mathcal{E}, \mathcal{E} \dashrightarrow \mathcal{E}_1)\}$

$\mathcal{F}\ell^b_{C,\mu} : \mathrm{Perf}_{/C^b} \longrightarrow \mathrm{Sets}$

$S \longmapsto \{(\mathcal{E}, \mathcal{E} \dashrightarrow \mathcal{E}_1) \mid \mathcal{E} \text{ fiberwise isom. to } \mathcal{E}_b\}$

<u>Sketch of proof</u> (⚠ Very vague) ① Pretend a six functor formalism for L coefficient

After replacing ρ by unramified twist, one can take $K^p \subset G'(\mathbb{A}^p_f)$, $\mathcal{L}_\xi$ algebraic rep'n of C' with

highest wt ξ s.t. ρ is a direct summand of $f_*\mathcal{L}_\xi = \mathcal{A}_{C'}(\underline{G'(\mathbb{Q})}\backslash G'(\mathbb{A}_f)/K^p, \mathcal{L}_\xi)$

$$\begin{tikzcd}
\mathcal{L}_\xi & \mathcal{L}_\xi & S(G,X)^b \arrow[d, phantom, "\shortparallel"] & \mathcal{L}_\xi \\
{[*/K_\ell]} & {[\underline{C(\mathbb{Q})}\backslash G'(\mathbb{A}_f)/K^p \cdot G_b(\mathbb{Q}_p)]} \arrow[l, "s"'] \arrow[d, "f"] & \left(\underline{C'(\mathbb{Q})}\backslash G'(\mathbb{A}_f)/K^p \times \mathrm{Sh}(G,\mu,b)_\infty\right)/\underline{G_b(\mathbb{Q}_p)} \arrow[l, "g"'] \arrow[r, hook] \arrow[d, "\pi^b_{HT}"] & S(G,X)_{K^p} \arrow[d, "\pi_{HT}"] \\
\rho & {[*/\underline{G_b(\mathbb{Q}_p)}]} & \mathcal{F}\ell^b_{C,\mu} \arrow[l, "q^b"'] \arrow[r, hook, "j"] & \mathcal{F}\ell_{C,\mu}
\end{tikzcd}$$

$R\Gamma_c(\mathrm{Sh}(G,\mu,b)_\infty, L)[\rho] \simeq R\Gamma_c(\mathcal{F}\ell^b_{C,\mu}, (q^b)^*\rho) \simeq R\Gamma(\mathcal{F}\ell_{C,\mu}, j_!(q^b)^*\rho)$

$X/G \xrightarrow{f} */C$, $p \downarrow *$ LHS $= \mathrm{Hom}(f_!L, \rho^*)^* = \mathrm{Hom}(L, f^!\rho^*)^*$

$= (p_* f^! \rho^*)^* = p_! f^*\rho =$ RHS

$R\Gamma(S(G,X)_{K^p}, \mathcal{L}_\xi) \simeq R\Gamma(\mathcal{F}\ell_{C,\mu}, (\pi_{HT})_*\mathcal{L}_\xi)$

② Pretend it's ℓ-coh sm.

$$\begin{tikzcd}
\mathcal{F}\ell^b_{C,\mu} \arrow[r, hook, "j"] \arrow[d, "q^b"'] & \mathcal{F}\ell_{C,\mu} \arrow[d, "q"] \\
\mathrm{Bun}^b_G \arrow[r, hook, "j_b"'] & \mathrm{Bun}_G
\end{tikzcd}$$

Want $j_!(q^b)^*\rho$ is a direct summand of $(\pi_{HT})_*\mathcal{L}_\xi$

This follows from $(q^b)^*\rho$ is a direct summand of $j^*(\pi_{HT})_*\mathcal{L}_\xi = (\pi^b_{HT})_* g^*\mathcal{L}_\xi = (q^b)^* f_*\mathcal{L}_\xi$

q ℓ-coh smooth

$j_!(q^b)^*\rho \simeq q^* j_{b,!}\rho \simeq q^* j_{b,*}\rho \simeq j_*(q^b)^*\rho$

Li-Schwermer $\Rightarrow$ $R\Gamma(S(G,X), \mathcal{L}_\xi)$ vanishes in degree $i < d = \dim S(G,X) = \dim Sh(G,\mu,b)_\infty$

<u>Thm 2.23</u> $(R\Gamma(Sh(G,\mu,b)_\infty, L)[\rho])^* \simeq R\Gamma(Sh(G,\mu,b)_\infty, L)[\rho^*][2d](d) \Rightarrow \checkmark$ $\square$.

<u>How to make this arguement mathematically correct</u>?

① Work with $\mathcal{O}_E$-coef for $E/\mathbb{Q}_l$ finite ext'n, invert l at last step

② $[Fl_{G,\mu}/\underline{K_p}] \longrightarrow Bun_G$ is l-coh sm. quotient K_p everywhere, take limit at last step.