LV trace formula for v-stacks

§1 About trace

k field, V f.d. k-v.s.

$$\begin{tikzcd} k \arrow[r] \arrow[rr, bend left, "\text{coev}"] & \mathrm{End}_k(V) & V \otimes_k V^\vee \arrow[l, "\sim"'] \arrow[r, "\text{ev}"] & k \end{tikzcd}$$

$$1 \longmapsto \mathrm{id}_V \qquad\qquad v \otimes f \longmapsto f(v)$$

$$\rightsquigarrow \quad k \xrightarrow{\text{coev}} V \otimes V^\vee \xrightarrow{u \otimes \text{id}} V \otimes V^\vee \longrightarrow k \quad \text{with } u: V \to V.$$

$$1 \longmapsto \mathrm{tr}(u).$$

Define $\mathcal{C} = (\mathcal{C}, \otimes, 1_{\mathcal{C}})$ symm monoidal 2-cat.

$V \in \mathcal{C}$ dualizable if $\exists V^\vee \in \mathcal{C}$ s.t. $\cdot$ $\text{ev}: V \otimes V^\vee \longrightarrow 1_{\mathcal{C}}$

$\cdot$ $\text{coev}: 1_{\mathcal{C}} \longrightarrow V^\vee \otimes V$.

$(V \xrightarrow{\text{id} \otimes \text{coev}} V \otimes V^\vee \otimes V \xrightarrow{\text{ev} \otimes \text{id}} V) \simeq \text{id}$.

$(V^\vee \xrightarrow{\text{coev} \otimes \text{id}} V^\vee \otimes V \otimes V^\vee \xrightarrow{\text{id} \otimes \text{ev}} V^\vee) \simeq \text{id}$.

- $(-) \otimes V \dashv (-) \otimes V^\vee \dashv (-) \otimes V$.
- $(V^\vee)^\vee = V$
- $V^\vee \simeq \underline{\mathcal{H}om}(V, 1_{\mathcal{C}})$ if $\underline{\mathcal{H}om}$ exists

Define for $u: V \longrightarrow V$ that

$$\mathrm{tr}(u): 1_{\mathcal{C}} \xrightarrow{\text{coev}} V \otimes V^\vee \xrightarrow{u \otimes \text{id}} V \otimes V^\vee \xrightarrow{\text{ev}} 1_{\mathcal{C}}$$

$\rightsquigarrow$ $\mathrm{tr}(u) \in \mathrm{End}(1_{\mathcal{C}}) = \Omega\mathcal{C}$.

& $\dim V = \mathrm{tr}(\mathrm{id}_V) \in \Omega\mathcal{C}$.

E.g. (1) k field, $\mathcal{C} = \mathrm{Vect}_k$.

V dualizable $\iff$ $\dim_k V < +\infty$.

(2) R comm ring, $\mathcal{C} = \mathrm{Mod}_R$

M dualizable $\iff$ M f.g. proj. R-mod.

$\Rightarrow M$ reflexive, i.e. $M \xrightarrow{\sim} \mathrm{Hom}_R(\mathrm{Hom}_R(M,R),R)$.

<u>But</u> reflexive $\not\Rightarrow$ dualizable:

$R = \mathbb{Z}$, $M = \bigoplus_{i=1}^{\infty} \mathbb{Z}$ reflexive but not dualizable.

free $\not\Rightarrow$ reflexive (not necessarily).

(3) X sch, $\mathcal{C} = D_{qcoh}(X)$ w/ $(-)\otimes^{L}(-)$.

M dualizable $\in \mathcal{C} \iff M$ perfect complex.

(4) $\mathrm{Cob}(n)$ Cobordism cat.

<u>Obj</u> closed oriented manifolds of dim $n-1$

<u>Bordism</u> $M \to N$, B oriented manifold of dim n.

$$\partial B = \bar{M} \sqcup N.$$

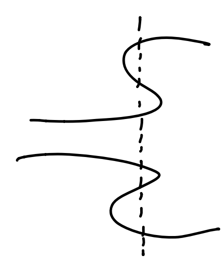
<u>Mor</u> diffeom classes of bordisms.

$\rightsquigarrow \mathrm{Cob}(1)$: <u>Obj</u> $+, -$ (orientations of pts).

<u>Mor</u> $- \bullet\!\!-\!\!\!-\!\!\bullet +$: $\mathrm{id}_{\{+\}}$

$^{+}_{-}\supset$: coev: $\phi \to \{+\}\sqcup\{-\}$

$\subset^{+}_{-}$: ev: $\{+\}\sqcup\{-\} \to \phi$.



<u>Fact</u> $\mathcal{C}$ symm monoidal cat:

$$\mathrm{Fun}^{\otimes}(\mathrm{Cob}(1), \mathcal{C}) \simeq \underbrace{(\mathcal{C}^{d})^{\simeq}}_{\text{taking Mor by kicking out all non-isoms.}}$$

<u>Category of correspondences</u>

$\mathcal{S}$ $\underbrace{(2,1)\text{-cat}}$ admitting fin limits

↑ w/ 2-mors are isoms.

$\mathrm{Corr}_{\mathcal{S}}$: $\mathrm{ob}(\mathrm{Corr}_{\mathcal{S}}) = \mathrm{ob}(\mathcal{S})$

<u>morph</u> $X \to Y$ via

$X \xleftarrow{\overleftarrow{c}} C \xrightarrow{\overrightarrow{c}} Y$, $c: C \to X \times Y$.

Composition of morph:

$$\begin{tikzcd} & & C\times_Y D \arrow[dl] \arrow[dr] & & \\ & C \arrow[dl] \arrow[dr] & & D \arrow[dl] \arrow[dr] & \\ X & & Y & & Z \end{tikzcd}$$

2-morph diagram
$$\begin{tikzcd} & C \arrow[dl, "\overleftarrow{c}"'] \arrow[dr, "\overrightarrow{c}"] \arrow[d] & \\ X \arrow[r, phantom, "\Updownarrow"] & D \arrow[l, "\overleftarrow{d}"] \arrow[r, "\overrightarrow{d}"'] \arrow[r, phantom, "\Updownarrow"] & Y \end{tikzcd}$$

$X \underset{\uparrow}{\otimes} Y = X \times Y$ (product in $\mathcal{S}$).

in $\mathrm{Corr}_{\mathcal{S}}$.

$(\mathrm{Corr}_{\mathcal{S}}, \times, e)$ symm monoidal 2-cat.

Fact Every obj $X \in \mathrm{Corr}_{\mathcal{S}}$ is dualizable ($X^\vee = X$).

Trace: ev:
$$\begin{tikzcd} & X \arrow[dl, "\Delta"'] \arrow[dr] & \\ X\times X & & e \end{tikzcd}$$
coev:
$$\begin{tikzcd} & X \arrow[dl] \arrow[dr, "\Delta"] & \\ e & & X\times X \end{tikzcd}$$
c:
$$\begin{tikzcd} & C \arrow[dl, "\overleftarrow{c}"'] \arrow[dr, "\overrightarrow{c}"] & \\ X & & X \end{tikzcd}$$

Then $\mathrm{tr}(c)$ is the diagram (or $\mathrm{Fix}(c)$ itself):
$$\begin{tikzcd} & & & \mathrm{Fix}(c) \arrow[dll, bend right] \arrow[drr, bend left] & & & \\ & X \arrow[dl] \arrow[dr, "\Delta"] & & C\times X \arrow[dl, "\overleftarrow{c}\times id"'] \arrow[dr, "\overrightarrow{c}\times id"] & & X \arrow[dl, "\Delta"'] \arrow[dr] & \\ e & & X\times X & & X\times X & & e \end{tikzcd}$$
$$\begin{tikzcd} \mathrm{Fix}(c) \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner"] & C \arrow[d, "c"] \\ X \arrow[r, "\Delta"] & X\times X \end{tikzcd}$$

Loop action: $\Omega \mathrm{Corr}_{\mathcal{S}} = \mathcal{S}$ via $(e \leftarrow C \rightarrow e) \leftarrow\!\shortmid C$.

Example Inertia obj:
$$\begin{tikzcd} \mathrm{In}(x) \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner"] & X \\ X \arrow[r, "\Delta"] & X\times X \arrow[u, "\Delta"'] \end{tikzcd}$$
$\rightsquigarrow \dim(x) = \mathrm{tr}(id_x)$.

When $\mathcal{S}$ = 1-cat. get $\mathrm{In}(x) = X$.

Important Eg. S diamond, $\mathcal{S}$ = v-stack$/S$, $G \to S$ groupoid.

$\rightsquigarrow [S/G] \in \mathcal{S}$ small v-stack.

$\Rightarrow \mathrm{In}([S/G]/S) = [G /\!/ G]$

action of G on G by conjugation.

§2 Sheaf theory

Consider the right-lax symm monoidal functor

$$\mathrm{Corr}_S^P \longrightarrow \mathrm{Cat}^{co} \leftarrow \text{reversed 2-morph}$$

$$X \longmapsto D(X)$$

$$\left(\begin{tikzcd} & C \arrow[dl, "\overleftarrow{c}"'] \arrow[dr, "\overrightarrow{c}"] & \\ X & & Y \end{tikzcd}\right) \longmapsto \left(\overrightarrow{c}_!\,\overleftarrow{c}^* : D(X) \to D(Y)\right).$$

$$\left(\begin{tikzcd} & & C\times D \arrow[dl] \arrow[dr] \arrow[dd, phantom, "\wedge"] & & \\ & C \arrow[dl, "\overleftarrow{c}"'] \arrow[dr, "\overrightarrow{c}"] & & D \arrow[dl, "\overleftarrow{d}"'] \arrow[dr, "\overrightarrow{d}"] & \\ X & & Y & & Z \end{tikzcd}\right) \longmapsto \text{Base change}$$

Define $D(X) \times D(Y) \xrightarrow{\boxtimes} D(X\times Y)$ with $\begin{tikzcd} X & X\times Y \arrow[l, "p_X"'] \arrow[r, "p_Y"] & Y \end{tikzcd}$

$$(A, B) \longmapsto A \boxtimes B := p_X^* A \otimes p_Y^* B$$

$$\begin{tikzcd} & C \arrow[dl, "\overleftarrow{c}"'] \arrow[dr, "\overrightarrow{c}"] \arrow[dd, "p"] & \\ X \arrow[r, phantom, "\Updownarrow"] & & Y \arrow[l, phantom, "\Updownarrow"] \\ & D \arrow[ul, "\overleftarrow{d}"] \arrow[ur, "\overrightarrow{d}"'] & \end{tikzcd} \qquad \begin{tikzcd} \overrightarrow{c}_!\,\overleftarrow{c}^* \\ \overrightarrow{d}_!\,\overleftarrow{d}^* \arrow[u] \end{tikzcd} \qquad \text{for } p \in P \text{ "proper".}$$

$$\begin{tikzcd} & C \arrow[dl, bend right, "p"'] \arrow[d] \arrow[dr, bend left, "p"] & \\ X \arrow[r, equal] & X \arrow[r, equal] & X \end{tikzcd} \qquad \begin{matrix} p_!p^* = p_*p^* \longleftarrow \mathrm{id} \\ (p \in P \Rightarrow p_! \simeq p_*.) \end{matrix}$$

<u>Examples</u> (1) Sch: · qcqs schemes

· Compactifiable (= sep + f.t.) morph.

$S \in \mathrm{Sch}$: $\mathcal{S} = \mathrm{Sch}/S$, $P = \{\text{proper morph}\}$, $n\Lambda = 0$ for n invertible in S.

$$\rightsquigarrow \left.\begin{matrix} \mathrm{Corr}_S^P \longrightarrow \mathrm{Cat}^{co} \\ X \longmapsto D_{et}(X, \Lambda). \end{matrix}\right\} \text{(Scholze)}.$$

(2) Diam: · locally spatial diamonds

· nice (= Compactifiable + locally $\dim.\mathrm{trg} < \infty$) morphisms.

$S \in \mathrm{Diam}$: $\mathcal{S} = \mathrm{Diam}/S$, $P = \{\text{proper morph}\}$, $n\Lambda = 0$.

$\rightsquigarrow$ $\mathrm{Corr}_S^P \longrightarrow \mathrm{Cat}^{co}$ as in (1)

(3) St : • descent v-stacks

• fine morphisms.

$S \in St$: $\mathcal{C} = St/S$, $P = \{\text{proper morph}\}$.

By Gulotta-Hansen-Weinstein:

$$\begin{tikzcd} \mathrm{Corr}_{\mathcal{C}} := \mathrm{Corr}_{\mathcal{C}}^{P} \arrow[r] & \mathrm{Cat}^{co} \\ X \arrow[r, mapsto] & D_{ét}(X, \Lambda). \end{tikzcd}$$

• A descent v-stack X is a small v-stack s.t.

(i) $\Delta : X \to X \times X$ representable by locally sep locally spatial diams

$$\begin{tikzcd} T' \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & T \arrow[d] \\ X \arrow[r] & X \times X \end{tikzcd}$$

T loc sep loc sp diam $\Rightarrow$ Same for T'.

(ii) (chart) $\exists\ U \xrightarrow{f} X$ surj, rep'ble in locally spatial diam

U loc sep loc sp diam.

locally on U, f is sep and $\underbrace{\text{coh sm}}_{\text{key input !}}$.

• $f : X \to Y$ morph of descent v-stacks is fine

if $\exists$

$$\begin{tikzcd} W \arrow[r, "g"] \arrow[d, "b"'] & V \arrow[d, "a"] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

a, b charts,

g nice (locally on W).

Let $\mathcal{B} = $ cat. $F : \mathcal{B} \to \mathrm{Cat}$ with cofibered cat

$$\begin{tikzcd} F(x) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathcal{F} \arrow[d] \\ x \arrow[r, "\in"] & \mathcal{B} \end{tikzcd}$$

Grothendieck constr:

$\mathcal{F}$: obj (X, A), $X \in \mathcal{B}$, $A \in F(x)$

mor $(f, w) : (X, A) \to (Y, B)$

where $f: X \to Y$, $u: F(f)A \to B$.

$$\begin{array}{ll} \mathrm{CoCorrs} & \underline{\text{obj}}\ (X, A),\ X \in \mathrm{St}/S,\ A \in D_{\text{et}}(X, \Lambda). \\ \big\downarrow & \underline{\text{mor}}\ (c, u) : (X, A) \longrightarrow (Y, B) \\ \mathrm{Corrs} & \quad \text{where } u : \overleftarrow{c}^* A \longrightarrow \overrightarrow{c}^! B \qquad (X \xleftarrow{\overleftarrow{c}} C \xrightarrow{\overrightarrow{c}} Y). \\ & \qquad\quad u^\# : \overrightarrow{c}_! \overleftarrow{c}^* A \longrightarrow B. \end{array}$$

Put $(X, A) \otimes (Y, B) = (X \times_S Y, A \boxtimes_S B)$

$\rightsquigarrow$ $(\mathrm{CoCorrs}, \otimes, (S, \Lambda))$ symm monoidal 2-cat.

Thm (Fargues-Scholze)

(X, A) dualizable in CoCorrs $\iff$ A ULA $/S$.

§3 LV trace

$\Omega \mathrm{CoCorrs}_S = \mathrm{End}((S, \Lambda)) = \{(X, \omega) \mid X \in \mathrm{St}/S,\ \omega \in H^0(X, K_{X/S})\}$

with $S \xleftarrow{\pi} X \xrightarrow{\pi} S$, $\omega : \underbrace{\pi^* \Lambda}_{\substack{\shortparallel \\ \Lambda}} \longrightarrow \pi^! \Lambda =: K_{X/S}$.

(X, A) dualizable, $(c, u) : (X, A) \longrightarrow (X, A)$.

Define the $\underline{\text{LV trace}}$ as

$$\mathrm{tr}(c, u) := (\mathrm{tr}(c), \mathrm{tr}(u)) \quad (\mathrm{tr}(c) = \mathrm{Fix}(c)).$$

with $\mathrm{tr}(u) \in H^0(\mathrm{Fix}(c), K_{\mathrm{Fix}(c)/S})$.

Functoriality of traces

$\mathcal{C}$ symm monoidal 2-cat.

$$\begin{tikzcd} X \arrow[d, bend right, "f"'] & \dim X \arrow[d] \\ Y \arrow[u, bend right, "g"'] & \dim Y \end{tikzcd} \quad \rightsquigarrow \quad \begin{tikzcd} & X \otimes X^\vee \arrow[dr, "\mathrm{ev}"] \arrow[dd] & \\ 1_{\mathcal{C}} \arrow[ur, "\mathrm{coev}"] \arrow[dr, "\mathrm{coev}"'] \arrow[r, phantom, "\Downarrow"] & & 1_{\mathcal{C}} \arrow[l, phantom, "\Downarrow"] \\ & Y \otimes Y^\vee \arrow[ur, "\mathrm{ev}"'] & \end{tikzcd}$$

The correct condition: $f \dashv g$ (f left adjoint of g).

$$fg \xRightarrow[\text{counit}]{\varepsilon} \mathrm{id} \qquad \mathrm{id} \xRightarrow[\text{unit}]{\eta} gf \qquad \rightsquigarrow \qquad \left.\begin{array}{l} f \Rightarrow fgf \Rightarrow f \\ g \Rightarrow gfg \Rightarrow g \end{array}\right\} \text{ are ids.}$$

$$\begin{tikzcd} X \arrow[r, "u"] \arrow[d, "f"'] \arrow[dr, phantom, "\Swarrow"] & X \arrow[d, "f"] \\ Y \arrow[r, "v"] & Y \end{tikzcd} \qquad f \dashv g \rightsquigarrow \begin{tikzcd} \mathrm{tr}(f) \arrow[d] \\ \mathrm{tr}(g) \end{tikzcd}$$

$$\begin{tikzcd} & X \otimes X^\vee \arrow[r, "u \otimes \mathrm{id}"] \arrow[d, "f \otimes g^\vee"] \arrow[dr, phantom, "\swarrow"] & X \otimes X^\vee \arrow[dr, "ev"] \arrow[d, "f \otimes g^\vee"] & \\ 1_{\mathcal{C}} \arrow[ur, "coev"] \arrow[dr, "coev"'] \arrow[r, phantom, "\Leftarrow"] & Y \otimes Y^\vee \arrow[r] & Y \otimes Y^\vee \arrow[r, phantom, "\swarrow"] \arrow[dr, phantom] & 1_{\mathcal{C}} \\ & Y \otimes Y^\vee \arrow[r] & Y \otimes Y^\vee \arrow[ur, "ev"'] & \end{tikzcd}$$

Corrs

$$\begin{tikzcd} & C \arrow[dl, "\overleftarrow{c}"'] \arrow[dr, "\overrightarrow{c}"] \arrow[dd, "p"] & \\ X & & Y \\ & D \arrow[ul, "\overleftarrow{d}"] \arrow[ur, "\overrightarrow{d}"'] & \end{tikzcd} \qquad p \text{ proper.}$$

ΩCorrs : St/S with proper morph only.

CoCorrs :

$$\begin{tikzcd} (c, u) \arrow[d] \\ (d, v) \end{tikzcd} \quad p : c \to d, \quad v = p_* u. \quad \rightsquigarrow \quad \begin{tikzcd} (X, \omega) \arrow[d] \\ (Y, \omega') \end{tikzcd} \quad p : X \to Y, \quad \omega' = p_* \omega.$$

$$\begin{tikzcd} H^0(X, K_{X/S}) \arrow[d, "p_*"] \\ H^0(Y, K_{Y/S}) \end{tikzcd} \quad \text{with } K_{X/S} = p^! K_{Y/S},$$

$$\rightsquigarrow \qquad p_* K_{X/S} = p_* p^! K_{Y/S} = p_! p^! K_{Y/S} \xrightarrow{\text{adj.}} K_{Y/S}.$$

Thm (LV trace formula).

$$\begin{tikzcd} X \arrow[d, "f"'] & C \arrow[l, "\overleftarrow{c}"'] \arrow[r, "\overrightarrow{c}"] \arrow[d, "p"] \arrow[dl, phantom, "\Nearrow"] \arrow[dr, phantom, "\Nwarrow"] & X \arrow[d, "f"] \\ Y & D \arrow[l, "\overleftarrow{d}"] \arrow[r, "\overrightarrow{d}"'] & Y \end{tikzcd}$$

(X, A) dualizable, $(c, u) : (X, A) \to (X, A)$

$u : \overleftarrow{c}^* A \to \overrightarrow{c}^! A$

with f & p proper.

$\Rightarrow (Y, f_* A)$ dualizable.

Hence $\mathrm{tr}(c, u) \longrightarrow \mathrm{tr}(d, p_* u)$ in $H^0(\mathrm{Fix}(d), K_{\mathrm{Fix}(d)/S})$. $(\mathrm{Fix}(c) \xrightarrow{q} \mathrm{Fix}(d))$

i.e. $\mathrm{tr}(\mathrm{id}_{(X,A)}) = (\mathrm{In}(X/S), cc_{X/S}(A))$.

Cor
$$\begin{tikzcd} X \arrow[d, "f"', "\text{proper}"] \\ Y \end{tikzcd}$$
(X, A) dualizable $\Rightarrow (Y, f_* A)$ dualizable & $cc_{Y/S}(f_* A) = \mathrm{In}(f/S)_* \, cc_{X/S}(A)$.

Cor $Y = D$, $S = \mathrm{Spd}(C)$, C alg closed perf'd field.
Then $\mathrm{tr}(u) = \mathrm{tr}(p_* u \mid R\Gamma(X, A))$.

E.g. $S = \mathrm{Spd}(C)$. $D_{\text{ét}}(S, \Lambda) = D(\mathrm{Mod}_\Lambda)$.
T locally profinite set, $T_S = \underline{T} \times S$ diamond.
$\rightsquigarrow D_{\text{ét}}(T_S, \Lambda) = D(\mathrm{Shv}(T, \Lambda))$
$\mathrm{Shv}(T, \Lambda) \simeq \mathrm{Mod}\, C(T, \Lambda)$
$\mathcal{F}_M \longleftarrow\!\shortmid M$ ($C(T, \Lambda)$-mod)
note M smooth $\Leftrightarrow M \otimes_{C(T,\Lambda)} C_c(T, \Lambda) \xrightarrow{\sim} M$.
where $\mathcal{F}_M(U) = \mathbb{1}_U \cdot M$ for U compact open.

Consider $f : T_S \to S$ nice,
then $f_! M = M$,
$f^! N = R\mathrm{Hom}_\Lambda(C_c(T, \Lambda), N)^{sm}$
$H^0(T_S, K_{T_S/S}) = \mathrm{Hom}(C_c(T, \Lambda), \Lambda) = \mathrm{Dist}(T, \Lambda)$.
Assume $\exists X$ sep loc sp diam & $X \to S$ surj, coh sm, with a free G_S-action.
Then: e.g. G locally pro-p gp
$[S/G_S]$ descent v-stack,
$f : [S/G_S] \to S$ fine, coh sm of dim 0.

$$\mathrm{Det}([S/G_S],\Lambda) = D(G,\Lambda) = D(\mathrm{Mod}^{sm}(G,\Lambda)).$$

$\rightsquigarrow$ $f^*N = (N,$ triv. G-action$)$.

$f_*M = M^G$ derived

$f^!N = \mathrm{Haar}(G,\Lambda)^* \otimes_\Lambda N$

$f_!M = (M \otimes_\Lambda \mathrm{Haar}(G,\Lambda))_G$ derived.

Lem $f: X \to S$ coh sm, $\Delta: X \longrightarrow X \times_S X$.

Then $\Delta^!\Lambda \otimes f^!\Lambda \simeq \Lambda_X$.

Proof

$$\begin{tikzcd}
 & X \arrow[d, "\Delta"] \arrow[ddl, bend right, "="'] \arrow[ddr, bend left, "="] & \\
 & X \times_S X \arrow[dl, "p_1"'] \arrow[dr, "p_2"] \arrow[d, phantom, "\wedge"] & \\
X \arrow[dr, "f"'] & & X \arrow[dl, "f"] \\
 & S &
\end{tikzcd}$$

$$\Lambda_X = \Delta^! p_1^! \Lambda = \Delta^!\Lambda \otimes \Delta^* \underbrace{p_1^!\Lambda}_{= p_2^* f^!\Lambda}$$

$$= \Delta^!\Lambda \otimes f^!\Lambda \qquad \square$$

Resume Consider $\Delta: [S/G_S] \longrightarrow [S/G_S] \times_S [S/G_S] = [S/(G\times G)_S]$

$\leftrightarrow \Delta: G \longrightarrow G \times G$

$\Rightarrow \Delta_! = c\mathrm{Ind}_{\Delta(G)}^{G\times G}$.

For $M = \mathrm{Haar}(G,\Lambda)$, we may assume $\Lambda = \mathbb{Z}/n\mathbb{Z}$.

$\Rightarrow \mathrm{Hom}_G(M, \Delta^!\Lambda) = \mathrm{Hom}_{G\times G}(\Delta_! M, \Lambda)$ $\quad \delta(g)$ = modular func.

b/c $\Delta_! M = \{ f: G\times G \to \Lambda \mid f(gg_1, gg_2) = \delta(g) f(g_1,g_2), \forall g_1, g_2, g \in G\}$.

$$\Delta_! M \longrightarrow \Lambda \quad \text{torsion-free}$$

$$f \longmapsto \int_{g\in G} f(g^{-1}, 1)\, d\mu(g)$$

(note $K_{[S/G_S]/S} = f^!\Lambda = \mathrm{Haar}(G,\Lambda)^*$.).

E.g. $G \curvearrowright T$, $X = [T_S / G_S]$,

$$H^0(X, K_X) = \mathrm{Hom}_G(C_c(T, \Lambda) \otimes \mathrm{Haar}(G, \Lambda), \Lambda)$$

$$\mathrm{In}([S/G_S]/S) = [G_S /\!/ G_S].$$

For $(X, A) = ([S/G_S], A)$,

$$cc_{[S/G_S]/S}(A) = H^0([S/G_S]/S, K_{[S/G_S]/S})$$

$$\| \quad \mathrm{Hom}_G(C_c(\underset{\circlearrowleft}{G}, \Lambda) \otimes \mathrm{Haar}(G, \Lambda), \Lambda)$$

G acts by conj.