Local terms of B_{dR}-affine Grassmannian

<u>Review on Sat: 3 approaches</u>

(1) $F/\mathbb{Q}_p$ fin ext'n, with $C=\hat{\bar{F}}$, G/F conn red gp.

$$\rightsquigarrow \underset{\overset{\|}{LG/L^+G}}{\mathrm{Gr}_G} = \mathrm{Gr}_G^{B_{dR}}/\mathrm{Spd}\,C : \{\text{aff perf'd space}/C\}^{op} \longrightarrow \text{Sets}$$

$$LG : (R,R^+) \longmapsto G(B_{dR}(R^\flat))$$

$$L^+G : (R,R^+) \longmapsto G(B_{dR}^+(R^\flat)).$$

with $\mathrm{Gr}_G = \varinjlim \underbrace{\mathrm{Gr}_{G,\le\mu}}_{\text{spatial diamond, proper}/\mathrm{Spd}\,C}$, where $\mu \in$ cochar of G.

Define $\mathcal{H}ck_G := [L^+G\backslash \mathrm{Gr}_G]$ Hecke stack.

$$\mathrm{Sat}_G(\Lambda)^{*} = \mathrm{Perv}^{ULA}_{\Lambda\text{-flat}}(\mathcal{H}ck_G/\mathrm{Spd}\,C) \cong \mathrm{Rep}(\hat{G},\Lambda)^{\circledast}.$$

$$\mathcal{S}_V \longleftrightarrow V$$

for $\Lambda \in \mathbb{Z}/\ell^n\mathbb{Z}[\sqrt{q}]$.

(2) K = DVF, res field $= k$ assumed to be alg closed, perfect

G/K reductive gp. $\mathcal{G}/\mathcal{O}_K$ sm aff gp sch.

$\mathcal{G}_\eta \simeq G$, $\mathrm{Gr}^W_{\mathcal{G}} = L^WG/L^{+,W}G : \{\text{perfect alg}/k\} \longrightarrow \text{Sets}$

$$L^WG : R \longmapsto \mathcal{G}(W_{\mathcal{O}_E}(R)[\tfrac{1}{p}]).$$

$$L^{+,W}G : R \longmapsto \mathcal{G}(\underbrace{W_{\mathcal{O}_E}(R)}_{\overset{\|}{W(R)\otimes_{W(k)}\mathcal{O}_E}}).$$

Take $\mathrm{Gr}^W_{\mathcal{G}} := \varinjlim \underbrace{\mathrm{Gr}^W_{\mathcal{G},\le\mu}}_{\text{sch. perfect, fin type, and proper}/k}$ (satisfying geom Satake equiv).

(3) $S = \mathrm{Spd}\,\mathcal{O}_C$, $\mathcal{G}/\mathcal{O}_F$ red gp.

$$\mathrm{Gr}^{BD}_{\mathcal{G}}(\underbrace{\mathrm{Spa}(R,R^+)}_{\overset{\|}{T}}) = \{(D,\mathcal{E},\varphi)\}/\sim$$

- $D \subseteq S(T)$ be a Cartier divisor over $\mathcal{Y}_T$.
- $\mathcal{E}|_{\mathcal{Y}_T \setminus D} \simeq \mathcal{E}^{\text{triv}}$ triv torsor, meromorphic outside D.

<u>Comparisons</u>

$$\begin{tikzcd}
(\mathrm{Gr}^{W}_{G})^{\diamond}_{T} \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near start] & \mathrm{Gr}^{BD}_{G} \arrow[d] & \mathrm{Gr}_{G} \arrow[l] \arrow[d] \arrow[dl, phantom, "\urcorner", very near start] \\
S = \mathrm{Spd}\, \bar{\mathbb{F}}_p \arrow[r] & S & \mathrm{Spa}\, C = \eta \arrow[l]
\end{tikzcd}$$

ULA assumption is essential while working $/S$.

$$\begin{tikzcd}
 & \mathrm{D\acute{e}t}(\mathrm{Gr}_G, \Lambda) \\
\mathrm{Rep}(\hat{G}, \Lambda) \arrow[ur] \arrow[r] \arrow[dr, bend right] & \mathrm{D\acute{e}t}(\mathrm{Gr}^{BD}_G, \Lambda) \arrow[u, "j^*"'] \arrow[d, "i^*"] \\
 & \mathrm{D\acute{e}t}(\mathrm{Gr}^{W}_G, \Lambda)
\end{tikzcd}$$

essential images of $\mathrm{Rep}(\hat{G}, \Lambda)$ are equivalent to each other through j^* & i^*.

$G_{rs} \overset{\text{open}}{\subseteq} G$ open subgp of regular semisimple elements

$\{ g \in G : C(g, G)^{\circ} \text{ max'l torus} \}$.

$G_{sr} \overset{\text{open}}{\subseteq} G$ open subgp of strongly regular semisimple elements

$\{ g \in G : C(g, G) \text{ connected max'l torus} \}$.

$\rightsquigarrow \ G_{\substack{sr \\ rs}}(F) =: G(F)_{\substack{sr \\ rs}}$.

We need to understand local terms on Gr_G.

<u>Prop</u> K^+ DVR. $k = \bar{k}$ res field. $K = \mathrm{Frac}\, K^+$.

G/K^+ reductive, $g \in G(K^+)$, $\bar{g} \in G(k)_{sr}$.

$T = C(g, G)$. $T \subseteq G$ induces

$$T(K)/T(K^+) \simeq (G(K)/G(K^+))^{g}$$

$(LG \circlearrowright \mathrm{Gr}_G \Rightarrow G(F) \circlearrowright \mathrm{Gr}_G)$.

Notation Fix $\hat{T} \leq \hat{G} \rightsquigarrow \mathrm{Gr}_G^{g} \longrightarrow X^*(\hat{T})$

$x \longmapsto \nu_x$ open schubert orbit cell of Gr_G.

Thm $V \in \mathrm{Rep}(\hat{G}) \rightsquigarrow \mathcal{S}_V / \mathrm{Gr}_G$, $g \in G(F)_{sr}$, $\forall x \in \mathrm{Gr}_G^{g}$.

$$\mathrm{loc}_x(g, \mathcal{S}_V) = (-1)^{\langle 2\rho, \nu_x \rangle} \mathrm{rank}\, V[\nu_x]$$

If so, then:

- Can enlarge $F \rightsquigarrow \mathcal{G}/\mathcal{O}_F$ split, $\exists g \in G(k^+)$, $\bar{g} \in G(k)_{sr}$
  g finite order, $(\mathrm{ord}(g), \mathrm{char}\, k) = 1$.
- $g, g' \rightsquigarrow \underset{x \longleftrightarrow x'}{\mathrm{Gr}_G^{g} \simeq \mathrm{Gr}_G^{g'}} \Rightarrow \nu_x = \nu_{x'} \Rightarrow \mathrm{loc}_x(g, \mathcal{S}_V) = \mathrm{loc}_{x'}(g', \mathcal{S}_V)$.
- Locally.

$$\begin{tikzcd}
\mathrm{Gr}_{\mathcal{G}}^{W} \arrow[r, "i"] \arrow[d] \arrow[dr, phantom, "\ulcorner"] & \mathrm{Gr}_{\mathcal{G}}^{BD} \arrow[d] \arrow[dr, phantom, "\top"] & \mathrm{Gr}_G \arrow[l, "j"'] \arrow[d] \\
\mathrm{Spd}\,\mathcal{O}_C = S \arrow[r] & S & \eta \arrow[l]
\end{tikzcd}$$

$\beta \simeq S \subseteq \mathrm{Gr}_{\mathcal{G}}^{BD, g} \xleftarrow{\sim} \mathrm{Gr}_T \simeq X_*(T)_S$. $(T = \mathrm{Cent}(g, \mathcal{G}))$.

Then $\mathrm{loc}_{\beta_S}(\mathrm{Gr}_{\mathcal{G}}^{W}, i^* \mathcal{S}_V) = \mathrm{loc}_{\beta_\eta}(\mathrm{Gr}_G, j^* \mathcal{S}_V) \in \Lambda$.

- Take $g \in \mathcal{G}(\mathcal{O}_F)$ as above. $\forall x \in \mathrm{Gr}_{\mathcal{G}}^{W, g}$ & $V \in \mathrm{Rep}(\hat{G})$,

$$\mathrm{loc}_x(g, \mathcal{S}_V) = (-1)^{\langle 2\rho, \nu_x \rangle} \mathrm{rank}\, V[\nu_x].$$

$T = \mathrm{Cent}(g, \mathcal{G})$, $\nu \in X_*(T)$, $\underset{\substack{\shortparallel \\ LU \cdot \varpi^{\nu} L^+ G}}{S_\nu} \subseteq \mathrm{Gr}_{\mathcal{G}}^{W}$ semi-infinite orbit. (Choose $\varpi \in \mathcal{O}_F$.)

$G \supset B \twoheadrightarrow T$, $\mathcal{U}$ unipotent radical of B.

$\mathrm{Gr}_G = \bigcup_{\nu \in X_*(T)} S_\nu$, $\overline{S_\nu} = \bigcup_{\nu' \leq \nu} S_{\nu'}$, X = fin union of closed Schbert cell $\supseteq \mathrm{Supp}\, \mathcal{S}_V$

$$\begin{tikzcd}
 & \mathrm{Gr}_B^{W} \arrow[dl, bend right, "\hat{i}"'] \arrow[dr, "p"] & \\
\mathrm{Gr}_G^{W} & & \mathrm{Gr}_T^{W} = X_*(T) \ni \nu
\end{tikzcd}$$

where $i(p^{-1}(\nu)) = S_\nu$.

$X_\nu = X \cap S_\nu$, $X_{\leq \nu} = X \cap S_{\leq \nu}$, $\partial X_{\leq \nu} = X_{\leq \nu} \setminus X_\nu$ are g-stable.

Fact (1) $R\Gamma_c(X_\nu, S_\nu) \simeq V[\nu][-d]$, $d = \langle 2\rho, \nu \rangle$.

(2) $X_\nu^g = \{x_\nu\}$, g-action is trivial

$\Rightarrow \mathrm{Tr}(g, R\Gamma_c(X_\nu, S_\nu)) = \chi_c(x_\nu, S_\nu)$

$$= (-1)^{\langle 2\rho, \nu \rangle} \operatorname{rank} V[\nu].$$

On the other hand,

$$\begin{aligned} \mathrm{LHS} &= \mathrm{tr}(g \mid R\Gamma(X_{\le\nu}, S_\nu)) - \mathrm{tr}(g \mid R\Gamma(\partial X_{\le\nu}, S_\nu)) \\ &= \sum_{\nu' \le \nu} \mathrm{loc}_{x_{\nu'}}(g, S_\nu|_{X_{\le\nu}}) - \sum_{\nu' < \nu} \mathrm{loc}_{x_{\nu'}}(g, S_\nu|_{\partial X_{\le\nu}}) \\ &= \mathrm{loc}_{x_\nu}(g, S_\nu) \quad (\text{using } \mathrm{loc}_x(g, X) = \mathrm{loc}_x(g, X|_Z)). \end{aligned}$$

Prop $\bar{k} = k/\mathbb{F}_p$, X perfectly finite type k-sch.

($X = \operatorname{Spec} A$, $A \supseteq A_0$ f.t. k-sch s.t. $A_0^{\mathrm{perf}} = A$.)

$g : X \to X$ automorphism, finite prime-to-p order.

$A \in D^b_{\mathrm{perf}}(X, \mathbb{Z}_\ell)$, $u : g^*A \to A \Rightarrow \forall g$-isolated fixed pt x,

$$\mathrm{loc}_x(g, A) = \mathrm{tr}(g \mid A_x).$$

In particular, $Z \subseteq X$ g-stable closed subsch.

$$\mathrm{loc}_x(g, A|_Z) = \mathrm{loc}_x(g, A).$$

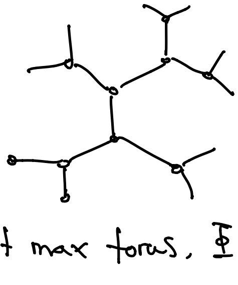
pf idea Deperfection to Varsharvsky's work.

Buildings Let G semisimple, split $/F$.

Hopefully, Bruhat-Tits building = simplicial complex = $B(G,F)$

$\circlearrowleft$ $G(F)$.

Picture



(likely : real Lie alg).

"glued up" by different apartments.

$T \subset G$ split max torus, $\Phi(G,T)$ roots.

Apartment $\mathcal{A}(G,T) = X_*(T)\otimes_{\mathbb{Z}}\mathbb{R}$ (e.g. SL_2).

$$\begin{array}{ll} \mathcal{B}(G,F) \text{ glue all } \mathcal{A}(G,T) & \left(\begin{array}{l} T \text{ max'l split torus} \\ \forall T, T', \ \exists g\in G(F),\ gTg^{-1}=T' \end{array}\right). \\ \quad \shortparallel & \\ G\times\mathcal{A}/\sim & \end{array}$$

$\forall (g,x)\sim(h,y)$, $\exists n\in N(T,G)$, s.t. $y = nx$ ($\Rightarrow g^{-1}hn\in G_x$).

Here $G_x \subseteq G(F)$ open cpt, maximal if x is a vertex.

$\rightsquigarrow G_x = \langle T(\mathcal{O}_F), U_\alpha(\mathfrak{m}^{-\lfloor\alpha(x)\rfloor})\rangle$, $\alpha\in\Phi$, $\mathfrak{m}$ max'l ideal of $\mathcal{O}_F$,

U_α root subgp.

- G_x acts transitively on the apartment through x.
- G acts transitively on all apartments.

Back to $(G(k)/G(k^+))^g = T(k)/T(k^+)$.

G_k semisimple split,

$G(k^+)\subset G(k)$ corresp to $\underset{\substack{\uparrow\\ \text{hyperspecial}}}{o}\in\mathcal{B}(G,F)$ & $G_o = G(k^+)$.

$G(k)/G(k^+) \longleftrightarrow G(k)$-orbits through o.

$x\in(G(k)/G(k^+))^g$, $x = ho$, $h\in G(k)$.

$\forall\alpha: T\to\mathbb{G}_m(\mathcal{U}_F)$ root, $\alpha(g)$ project from $L\mathbb{G}_m\to\mathbb{G}_m$ is nonzero.

$\underset{\text{Fact}}{\Longrightarrow}$ $ho\in\mathcal{A}$ is an apartment of T

$o\in h^{-1}\mathcal{A}$, $o\in\mathcal{A}$, $G(k^+)$ acts transitively on apartments through o.

o hyperspecial, $G(k^+)$ realizes all Weyl reflections $\Rightarrow h\in T(k)$.

Local term and base change

X/S, $\mathcal{F}\in D^{ULA}_{ét}(X/S)$, $u\in \mathrm{Hom}(c_1^*\mathcal{F}, c_2^!\mathcal{F})$

$c : C \xrightarrow{(c_1, c_2)} X \times_S X \rightsquigarrow$ categorical trace $\mathrm{tr}_c(\mathcal{U}, \mathcal{F})$.

Take

$$\begin{tikzcd} (\mathrm{Fix}(c), \mathrm{tr}_c(\mathcal{U},\mathcal{F})) \arrow[dd, mapsto] & \in H^0(\mathrm{Fix}(c), K_{\mathrm{Fix}(c)/S}). \arrow[d] \\ & H^0(\beta, K_{\beta/S}) \arrow[d] \\ \mathrm{loc}_\beta(\mathcal{U},\mathcal{F}) \in H^0(S,\Lambda) & \in H^0(S,\Lambda) \end{tikzcd}$$

$$\begin{tikzcd} \mathrm{Fix}\, C \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near start] & C \arrow[d, "c"] \\ X \arrow[r, "\Delta"] & X \times X \end{tikzcd}$$

, $\beta \subset \mathrm{Fix}(c)$ open & closed.

Suppose $T \to S$. Given $\mathrm{Fix}(c)_T = \mathrm{Fix}(C_T)$.

$$\begin{tikzcd} X_T \arrow[d, "a_X"'] & C_T \arrow[l, "c_{1,T}"'] \arrow[r, "c_{2,T}"] \arrow[d, "a_C"'] \arrow[dr, phantom, "\ulcorner", very near start] & X_T \arrow[d] \\ X & C \arrow[l, "c_1"'] \arrow[r, "c_2"] & X \end{tikzcd}$$

Define $\mathcal{U}_T : c_{1,T}^* a_X^* \mathcal{F} \xrightarrow{\sim} a_C^* c_1^* \mathcal{F} \xrightarrow{a_C^* u} a_C^* c_2^! \mathcal{F} \xrightarrow{\mathrm{loc}} c_2^! a_T^* \mathcal{F}$.

$\Rightarrow (a_T^* \mathcal{F}, \mathcal{U}_T) \in \mathrm{End}(\mathrm{CoCorr}_T)$.

$\overset{\shortparallel}{\mathcal{F}_T}$

<u>Prop</u> $\beta \subset \mathrm{Fix}(c)$, $H^0(S, \Lambda) \longrightarrow H^0(T, \Lambda)$ <u>Key</u> $\mathrm{CoCorr}_S \xrightarrow{(-) \times_S T} \mathrm{CoCorr}_T$

sends $\mathrm{loc}_\beta(\mathcal{U}, \mathcal{F}) \longrightarrow \mathrm{loc}_{\beta_T}(\mathcal{U}_T, \mathcal{F}_T)$.

($\Rightarrow$ If S connected, $\beta \simeq S$, $\mathrm{loc}_\beta(\mathcal{U}, \mathcal{F}) \in H^0(S, \Lambda) = \Lambda$.
constant family w.r.t. S)

Consider X/S nice diamond $\circlearrowleft$ G/S cohom sm gp diamond.

$Y = [X/G] \leftarrow X$, $X \times G \to X$ via $(x, g) \mapsto g(x)$.

(1) $(Y, A) \in$ dualizable obj in $\mathrm{CoCorr}_S \rightsquigarrow cc_{Y/S}(A) \in H^0(\mathrm{In}_S(Y), K_{\mathrm{In}_S(Y)/S})$.

(2) base change to (X, A_X) dualizable. $u_g : A_X \to g^* A_X \simeq g^! A_X$.

$$\begin{tikzcd}
\mathrm{Fix}(g) \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near start] & X \arrow[d, "{(id, g)}"] \\
X \arrow[r, "\Delta"] & X \times X
\end{tikzcd}$$

$\rightsquigarrow \mathrm{tr}(g, \mathcal{U}_g) \in H^0(\mathrm{Fix}(g), K_{\mathrm{Fix}(g)/S})$.

For $c = pr \times \alpha : X \times_S G \longrightarrow X \times_S X$ via $(x, g) \longmapsto (x, g(x))$,

$\mathrm{Fix}(c) \subset X \times_S G \circlearrowleft G$, $h(x, g) = (hx, hgh^{-1})$ for $h \in G$.

For $\mathrm{Ins}_S(Y) = [\mathrm{Fix}(c)/G]$,

$\mathrm{Ins}_S(Y) \longrightarrow [S/G] \simeq [G/\!/G]$.

G equiv of $A|_X$, $\tilde{u} : A|_{X \times G} \longrightarrow \alpha^* A|_X$.

$$\begin{tikzcd}
\tilde{c}: & X \times G \arrow[dl, "\tilde{G} = id"'] \arrow[dr, "\tilde{\alpha} : (x,g) \mapsto (g(x), g)"] & \\
X \times G & & X \times G
\end{tikzcd}$$

$A|_{X \times G} \xrightarrow{u} \tilde{\alpha}^* A|_X = \tilde{\alpha}^! A|_X$.

$\rightsquigarrow (\mathrm{Fix}(\tilde{c}) \simeq (\mathrm{Fix}(c), A|_{X \times G})) \in \mathrm{CoCorr}_G$.

$\rightsquigarrow \mathrm{tr}_c(u, A|_{X \times G})$ s.t. $\forall g \in G$, $\mathrm{tr}_{c|_g}(u|_g, A|_{X \times \{g\}}) = \mathrm{tr}(g, \mathcal{U}_g)$

$$\begin{tikzcd}
\mathrm{Fix}(\tilde{c}) \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near start] & G \arrow[d] \\
\mathrm{Ins}_S(Y) \arrow[r] & {[G/\!/G]}
\end{tikzcd}$$

$\rightsquigarrow \iota : H^0(\mathrm{Ins}_S(Y), K_{\mathrm{Ins}_S(Y)/S}) \longrightarrow H^0(\mathrm{Fix}(\tilde{c}), K_{\mathrm{Fix}(\tilde{c})})$.

Thm The characteristic class $cc_{Y/S}(A) = \mathrm{tr}(u, A|_{X \times G})$

($\Rightarrow$ tr is indep of g.)

Under stable conditions,

$\mathrm{Tr}_c(\mathcal{U}, \mathcal{F}) = \mathrm{Tr}_{c|_Z}(\mathcal{U}|_Z, \mathcal{F}|_Z)$ (fixed pts $\subseteq Z$).

where Z = c-inv closed subsch of X $(c : C \longrightarrow X \times X)$

$\exists$ family of morphisms $\begin{cases} \text{for generic } Z \hookrightarrow X \\ \text{for special fiber } Z \hookrightarrow N_Z(X). \end{cases}$

$$\begin{tikzcd}
Z \arrow[d] & c^{-1}(Z \times Z) \arrow[l] \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near start] & Z \arrow[d] \\
X & C \arrow[l] \arrow[r] & X
\end{tikzcd}$$

Deformation to normal cone :

$$Z \hookrightarrow N_Z(X) = \mathrm{Spec}\left(\bigoplus_{i=1}^{\infty} \mathcal{I}_Z^n / \mathcal{I}_Z^{n+1}\right).$$

Replace X by $N_Z(X)$ & $\mathcal{F}$ by "$Sp(\mathcal{F})$".

$\rightsquigarrow$ $X = N_Z(X) \circlearrowleft \mathbb{A}^1$, $\mathcal{F} = Sp\mathcal{F} \circlearrowleft \mathbb{G}_m$ (weakly equiv.)

$$\mu_{\mathbb{G}_m} : X \times \mathbb{G}_m \longrightarrow X, \quad \mu_{\mathbb{G}_m}^* \mathcal{F} \longrightarrow \mathcal{F}_{\mathbb{G}_m} = \mathcal{F} \boxtimes \Lambda_{\mathbb{G}_m}.$$

By additivity of trace $\Rightarrow$ reduce to $\mathcal{F}|_Z = 0$

$\rightsquigarrow$ to prove $\mathrm{tr}_c(u, \mathcal{F}) = 0$.

$\forall c : C \longrightarrow X \times X$, $X \times \mathbb{A}^1 \xrightarrow{\mu} X$

extends to $c_{\mathbb{A}^1} : C_{\mathbb{A}^1} \longrightarrow X_{\mathbb{A}^1} \times_{\mathbb{A}^1} X_{\mathbb{A}^1}$, $\mathcal{F}_{\mathbb{A}^1} = \mathcal{F} \boxtimes \Lambda_{\mathbb{A}^1}$, $u_{\mathbb{A}^1}$.

$$(c, a) \longmapsto (a c_1(c), c_2(c), a)$$

s.t. $u_t \hookrightarrow X \times X$ $(t \in \mathbb{A}^1(k))$. $(u_1 = u)$.

We obtain the sequence

$$c_{\mathbb{A}^1,1}^{*} \mathcal{F}_{\mathbb{A}^1} = (c_1 \times \mathrm{Id}_{\mathbb{A}^1})^*(\mu^* \mathcal{F}) \xrightarrow{\nu} (c_1 \times \mathrm{Id}_{\mathbb{A}^1})^* \mathcal{F}_{\mathbb{A}^1} = c_1^* \mathcal{F} \boxtimes \Lambda$$
$$\longrightarrow c_2^! \mathcal{F} \boxtimes \Lambda = c_{2,\mathbb{A}^1}^! \mathcal{F}_{\mathbb{A}^1}.$$

For $\mu : X \times \mathbb{A}^1 \longrightarrow X$, $\mu_0 : X \longrightarrow Z \hookrightarrow X$,

$\mu_0^* \mathcal{F} = 0 \Rightarrow \mathcal{F}$ extends to $\mathbb{A}^1$ (weakly equiv)

$$\left.\begin{array}{l} \mathrm{tr}_{c_{\mathbb{A}^1}}(u_{\mathbb{A}^1}, \mathcal{F}_{\mathbb{A}^1}), \quad t = 0 : \mathrm{tr} = 0 \\ \qquad\qquad\qquad\qquad\quad\; t = 1 : \overset{\shortparallel}{\mathrm{tr}_c}(u, \mathcal{F}) \end{array}\right\} \text{since } \mathbb{A}^1 \text{ geom conn.}$$