

Application to Hecke stacks (I)

§1 Hecke stacks

F/\mathbb{Q}_p , \check{F} complete max unram.

C/\check{F} complete alg closed.

T/C perf'd space \hookrightarrow FF curve $X_T (= X_T^b)$
 \uparrow
 \mathcal{D}_T div of deg 1.

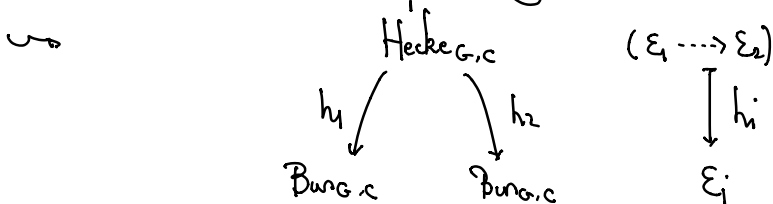
$\text{Bun}_{G,C} : \text{Perf}_C \rightarrow \text{Gpd}$.

$b \in G(\check{F}) \hookrightarrow \mathcal{E}^b$ ($b=1 \hookrightarrow \mathcal{E}^1$).

$\text{Hecke}_{G,C}$ = the v-stack on Perf_C , classifying morphisms

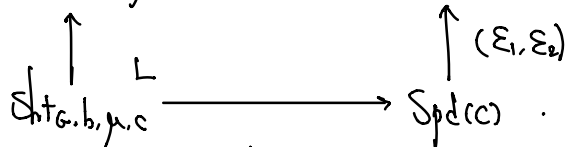
$$f: \mathcal{E}_1|_{X, \mathcal{D}} \xrightarrow{\sim} \mathcal{E}_2|_{X, \mathcal{D}}$$

which is meromorphic along \mathcal{D} .



Remark \exists bounded version of Hecke stacks with " $\leq \mu$ ".

Define $\text{Hecke}_{G, \leq \mu, C} \xrightarrow{h_1 = h_2} \text{Bun}_{G,C} = \text{Bun}_{G,C}$

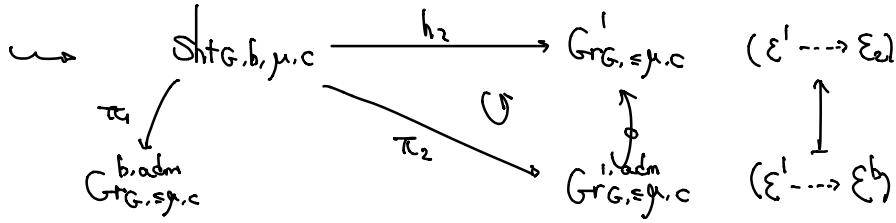


" $\{ \mathcal{E}^1 \dashrightarrow \mathcal{E}^b \mid \text{meromorphic along } \mathcal{D}, \hookrightarrow X, \text{ bounded by } \mu \}$ ".

Period map $b \in \mathcal{B}(G, \mu)_{\text{bas}}$,

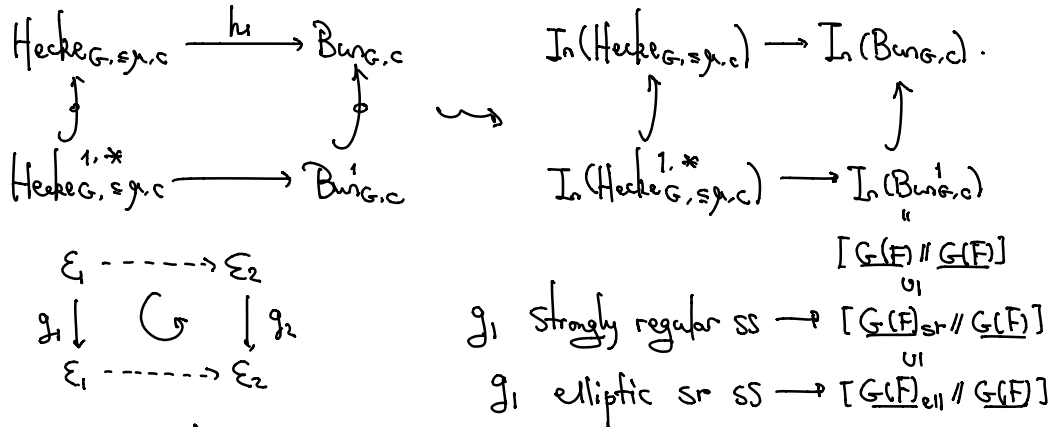
$\text{Sht}_{G,b,\mu,C} \xrightarrow{h_2} G_{G, \leq \mu, C}^1 = \{ \mathcal{E}^1 \dashrightarrow \mathcal{E}_2 \text{ on } X_0 \mid \text{mero \& bounded} \}$

$$(\mathcal{E}^1 \dashrightarrow \mathcal{E}^b) \longmapsto (\mathcal{E}^1 \dashrightarrow \mathcal{E}_2)$$



§2 Inertia stack of Hecke stacks

Taking inertia



Let $\mathcal{E}^1 \dashrightarrow \mathcal{E}^b$ be an object of $\text{In}(\text{Hecke}_{G,\mu,c})(c)$

$$\begin{array}{ccc}
 \mathcal{I} \downarrow & & \downarrow \mathcal{I}' \\
 \mathcal{E}^1 \dashrightarrow \mathcal{E}^b & & \text{with } (g, g') \in G(\mathbb{F})_{\text{sr}} \times G_b(\mathbb{F})_{\text{sr}}
 \end{array}$$

$$\downarrow \\
 \pi_2(x) \in Gr_{G,\mu,c}^{1,g} \text{ (g-fixed pt)}$$

$$\pi_1(x) \in Gr_{G,\mu,c}^{b,g'}$$

$$\left(\begin{array}{l}
 \text{Take } T = \text{Cent}(g, G) \hookrightarrow Gr_{T,\mu,c}^1 \hookrightarrow Gr_{G,\mu,c}^{1,g} \\
 \& G \text{ sr ss} \Rightarrow Gr_{T,\mu,c}^1 = Gr_{G,\mu,c}^{1,g} \\
 \lambda_{\mu}(T) \leq \mu
 \end{array} \right)$$

$$\Rightarrow \pi_2(x) \longleftrightarrow \lambda \in \lambda_{\mu}(T) \leq \mu, \quad \pi_1(x) \longleftrightarrow \lambda' \in \lambda_{\mu}(T') \leq -\mu.$$

$$T = \text{Cent}(g'; G).$$

Recall for each b , $G_{b,\mathbb{F}} \hookrightarrow G_{\mathbb{F}}$.

Prop g & g' are related, i.e. g & g' are conjugate in $G(\check{F})$ (hence in $G(\check{F})$).

Moreover, $\exists y \in G(\check{F})$ s.t. $g' = ygy^{-1}$, $\text{ad}(y): G_{\check{F}} \rightarrow G_{\check{F}}$

induces an F -rational isom $T_{\check{F}} \rightarrow T_{\check{F}}$

s.t. $g \mapsto g', \lambda \mapsto -\lambda'$.

Proof Consider
$$\begin{array}{ccccc} \mathcal{E}'|_{X_c|D_c} & \xrightarrow{x} & \mathcal{E}'|_{X_c|D_c} & \simeq & \mathcal{E}'|_{X_c|D_c} \\ g \downarrow & & \downarrow g' & & \downarrow g'' \\ \mathcal{E}'|_{X_c|D_c} & \xrightarrow{x} & \mathcal{E}^b|_{X_c|D_c} & \simeq & \mathcal{E}'|_{X_c|D_c} \end{array}$$

where $g'' =$ the image of g' along $G_b(F) \rightarrow G(\check{F})$.

$\Rightarrow g$ & g' conjugate in $G(B_c)$, $B_c = \mathcal{O}_X(X_c|D_c)$

Let $y \in G(\check{F})$ s.t. $(\text{ad}(y))(g) = g'$

$\Rightarrow (\text{ad}(y))(T_{\check{F}}) = T_{\check{F}}$.

$\Rightarrow \text{ad}(y): T_{\check{F}} \rightarrow T_{\check{F}}$, \check{F} -isom.

Claim $\text{ad}(y)$ is F -rational

- Galois action on $T_{\check{F}}$ is the standard one.

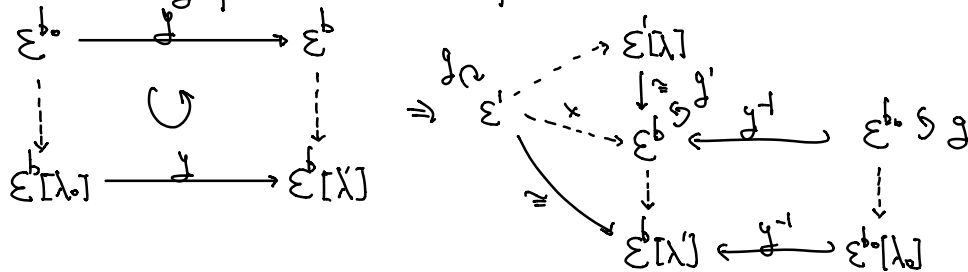
- Galois action on $T_{\check{F}} \in G_{\check{F}}$ is the twisted one.

$\sigma \in \text{Aut}(\check{F}/F)$ Frobenius, acting on $T_{\check{F}} = \text{int}(b) \circ \sigma$

Let $b_0 := y^{-1} b \sigma(y) \in T(\check{F})$ ($\Leftarrow g \in G(F), g' \in G_b(F)$.)

$\lambda_0 := \text{ad}(y^{-1})(\lambda) \in \chi_*(T)$.

The element y provides an isom of G -bundles:



$y^{-1}x$ comes from a modification of T -bundles.

$$\kappa(\varepsilon^{b_0}) = \kappa(\varepsilon^1) + \text{the class of } \lambda \text{ in } \pi_1(\mathbb{T})_\Gamma = \chi_*(\mathbb{T}).$$

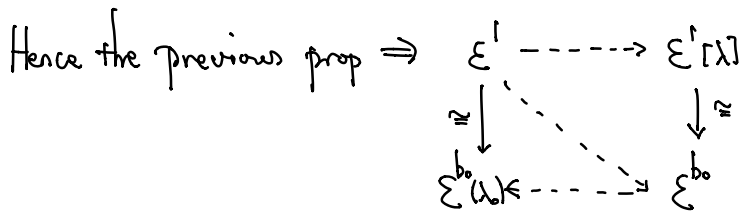
$$\Rightarrow \lambda = -\lambda_0 \text{ in } \chi_*(\mathbb{T}).$$

Def'n $g \in G(F)_{sr}, g' \in G_b(F)_{sr}$ related.
 let $y \in G(\check{F}), g' = ygy^{-1}, T = \text{Cent}(g, G)$.
 $\Rightarrow b_0 = y^{-1}by^0 \in T(\check{F})$.
 Put $\text{inv}[b](g, g') = [b_0] \in \mathcal{B}(\mathbb{T})$.

Fact $\forall z \in G(F), z' \in G_b(F)$.
 $\text{inv}[b](\text{ad}(z)(g), \text{ad}(z')(g')) = \text{ad}(z)(\underbrace{\text{inv}[b](g, g')}_{\in \mathcal{B}(\mathbb{T})}) \in \mathcal{B}(\text{ad}(z)(\mathbb{T})).$

Def'n $\text{Rel}_{b, (\varepsilon, \mu)} := \left\{ \begin{array}{l} (g, g', \lambda) \in G(F)_{sr} \times G_b(F)_{sr} \times \chi_*(G): \\ \text{(i) } g, g' \text{ related,} \\ \text{(ii) } T = \text{Cent}(g, G), \lambda \in \chi_*(\mathbb{T})_{(\varepsilon, \mu)} \\ \text{(iii) } \kappa(\text{inv}[b](g, g')) = \text{class of } \lambda \end{array} \right\} / \approx.$

where $(g, g', \lambda) \equiv (\text{ad}(z)(g), \text{ad}(z')(g'), \text{ad}(z)(\lambda)) \text{ modulo } \approx.$



$$\Rightarrow b_0 = y^{-1}by^0, \text{inv}[b](g, g') = [b_0] \in \mathcal{B}(\mathbb{T}).$$

$$\kappa(b_0) = \text{class of } \lambda \text{ in } \chi_*(\mathbb{T})_\Gamma.$$

$$\Rightarrow (g, g', \lambda) \in \text{Rel}_{b, (\varepsilon, \mu)}.$$

In particular, get a map

$$|\text{In}(\text{Hecke}_{G, (\varepsilon, \mu, c)}^{I, b})| \xrightarrow{\sim} \text{Rel}_{b, \mu, \text{ell}}$$

via
$$\left(\begin{array}{ccc} \mathcal{E}^1 & \dashrightarrow & \mathcal{E}^b \\ g \downarrow & & \downarrow g' \\ \mathcal{E}^1 & \dashrightarrow & \mathcal{E}^b \end{array} \right) \mapsto (g, g', \lambda).$$

Thm Assume $[b] \in \mathcal{B}(G, \mu)_{\text{bas}}$, $g \in G(F)_{\text{ell}}$, $g' \in G_b(F)_{\text{ell}}$
 $\lambda \in \chi_*(T) \in \mu$, $T = \text{Cent}(g, G)$.

Then $\mathcal{E}^1[\lambda] \cong \mathcal{E}^b$, and hence

$$G_{G, \text{syn}}^{1, g} \cong G_{G, \text{syn}}^{1, \text{adm}}$$

Similarly, $G_{G, \text{syn}}^{b, g'} \cong G_{G, \text{syn}}^{b, \text{adm}}$.

Claim $\mathcal{E}^1[\lambda]$ is semisimple.

pf. $\mathcal{E}^1[\lambda]$ comes from a T-bundle.

So only need to check
$$\begin{array}{ccc} \mathcal{B}(T) & \longrightarrow & \mathcal{B}(G) \\ & \searrow & \uparrow \\ & & \mathcal{B}(G)_{\text{bas}} \end{array}$$

Let $b_0 \in \mathcal{B}(T)$, $\mathcal{E}^{b_0} \rightarrow \nu_{b_0}: \mathcal{D} \rightarrow T/F$ Newton vector.

$$\begin{array}{ccc} \mathcal{D} & \longrightarrow & T/F \\ & \searrow & \uparrow \\ & & \mathcal{Z}(G^0) \end{array}$$

Cor $b \in \mathcal{B}(G, \mu)$. $|\text{In}(\text{Hecke}_{G, \text{syn}}^{1, b})_{\text{ell}}| \cong |\text{Rel}_{b, \mu, \text{ell}}|$.

§3 Transfer

\cdot b basic, $\text{Rel}_b \subseteq (G(F)_{\text{sr}} \times G_b(F)_{\text{sr}} \times \chi_*(G)) / G(F) \times G_b(F)$.

$$\begin{array}{ccc} & \swarrow & \searrow \\ G(F)_{\text{sr}} // G(F) & & G_b(F)_{\text{sr}} // G_b(F) \end{array}$$

Define $T_{b, \mu}^{G \rightarrow G_b}: \mathcal{Z}_c(G(F)_{\text{sr}} // G(F), \lambda) \longrightarrow \mathcal{Z}_c(G_b(F)_{\text{sr}} // G_b(F), \lambda)$.

$$T_{b,\mu}^{G \rightarrow G_b}(f)(g) := (-1)^d \sum_{(g', \lambda) \in \text{Rel}_b} f(g') \cdot \dim \gamma_{\mu}[\lambda]. \quad d = \langle 2\mu, 2\rho \rangle.$$

Also define $T_{b,\mu}^{G_b \rightarrow G}$ similarly. $t_{\mu} = \text{Weyl module}$.

$$C_c(G(F)_{\text{sr}} // G(F), \Lambda) \xleftarrow{\sim} C_c(G(F)_{\text{sr}}, \Lambda)_{G(F)}$$

$$\phi_G \longleftarrow \longrightarrow \phi$$

$$\begin{array}{ccc} \Rightarrow T_{b,\mu}^{G \rightarrow G_b} : C_c(G(F)_{\text{sr}}, \Lambda)_{G(F)} & \longrightarrow & C_c(G_b(F)_{\text{sr}}, \Lambda)_{G_b(F)} \\ & \uparrow & \uparrow \\ & C_c(G(F)_{\text{rel}}, \Lambda)_{G(F)} & \xrightarrow[\substack{q_{b,\mu}^{G \rightarrow G_b}}]{} C_c(G_b(F)_{\text{rel}}, \Lambda)_{G_b(F)} \end{array}$$

Def'n $\mathcal{T}_{b,\mu}^{G_b \rightarrow G} : \text{Dist}(G_b(F)_{\text{rel}}, \Lambda) \xrightarrow{G_b(F)} \text{Dist}(G(F)_{\text{rel}}, \Lambda)^{G(F)}$
is the Λ -dual of $T_{b,\mu}^{G \rightarrow G_b}$.

Prop $\Lambda = \overline{\mathbb{Q}_\ell}$, $\mathcal{T}_{b,\mu}^{G_b \rightarrow G}$ extends the transfer $T_{b,\mu}^{G_b \rightarrow G}$ in [HKW] §3.