Application to Hecke stacks (I)

§1 Hecke stacks

$F/\mathbb{Q}_p$. $\breve{F}$ complete max unram.

$C/\breve{F}$ complete alg closed.

T/C perf'd space $\rightsquigarrow$ FF curve X_T $(= X_{T^\flat})$ $\hookleftarrow$ D_T div of deg 1.

$\mathrm{Bun}_{G,C} : \mathrm{Perf}_C \longrightarrow \mathrm{Gpd}$.

$b \in G(\breve{F}) \rightsquigarrow \mathcal{E}^b$ $\quad (b = 1 \rightsquigarrow \mathcal{E}^1)$.

$\mathrm{Hecke}_{G,C}$ = the v-stack on Perf_C, classifying morphisms

$$f : \mathcal{E}_1|_{X_\bullet \setminus D_\bullet} \xrightarrow{\sim} \mathcal{E}_2|_{X_\bullet \setminus D_\bullet}$$

which is meromorphic along D.

$$\rightsquigarrow \quad \begin{tikzcd} & \mathrm{Hecke}_{G,C} \arrow[dl, "h_1"'] \arrow[dr, "h_2"] & \\ \mathrm{Bun}_{G,C} & & \mathrm{Bun}_{G,C} \end{tikzcd} \qquad \begin{tikzcd} (\mathcal{E}_1 \dashrightarrow \mathcal{E}_2) \arrow[d, mapsto, "h_i"] \\ \mathcal{E}_i \end{tikzcd}$$

<u>Rmk</u> $\exists$ bounded version of Hecke stacks with "$\le \mu$".

Define
$$\begin{tikzcd} \mathrm{Hecke}_{G,\le\mu,C} \arrow[r, "h_1 \times h_2"] & \mathrm{Bun}_{G,C} \times \mathrm{Bun}_{G,C} \\ \mathrm{Sht}_{G,b,\mu,C} \arrow[u] \arrow[r] \arrow[ur, phantom, "\lrcorner"] & \mathrm{Spd}(C) \arrow[u, "(\mathcal{E}_1, \mathcal{E}_2)"'] \end{tikzcd}$$

$\mathrm{Sht}_{G,b,\mu,C} = \{\mathcal{E}^1 \dashrightarrow \mathcal{E}^b \mid \text{meromorphic along } D_\bullet \hookrightarrow X_\bullet, \text{ bounded by } \mu\}$.

<u>Period map</u> $b \in B(G,\mu)_{\mathrm{bas}}$,

$$\mathrm{Sht}_{G,b,\mu,C} \xrightarrow{h_2} \mathrm{Gr}^1_{G,\le\mu,C} = \{\mathcal{E}^1 \dashrightarrow \mathcal{E}_2 \text{ on } X_\bullet \mid \text{mero \& bounded}\}$$

$$(\mathcal{E}^1 \dashrightarrow \mathcal{E}^b) \longmapsto (\mathcal{E}^1 \dashrightarrow \mathcal{E}_2)$$

$$\rightsquigarrow \begin{tikzcd} & \mathrm{Sht}_{G,b,\mu,C} \arrow[r, "h_2"] \arrow[dl, "\pi_1"'] \arrow[drr, "\pi_2"'] & & \mathrm{Gr}^{1}_{G,\le\mu,C} \arrow[d, phantom, ""] & (\mathcal{E}^1 \dashrightarrow \mathcal{E}_2) \\ \mathrm{Gr}^{b,adm}_{G,\le\mu,C} & & \circlearrowright & \mathrm{Gr}^{1,adm}_{G,\le\mu,C} \arrow[u, hook] & (\mathcal{E}^1 \dashrightarrow \mathcal{E}^b) \arrow[u, mapsto] \end{tikzcd}$$

§2 Inertia stack of Hecke stacks

Taking inertia

$$\begin{tikzcd} \mathrm{Hecke}_{G,\le\mu,C} \arrow[r, "h_1"] & \mathrm{Bun}_{G,C} \\ \mathrm{Hecke}^{1,*}_{G,\le\mu,C} \arrow[r] \arrow[u, hook] & \mathrm{Bun}^{1}_{G,C} \arrow[u, hook] \end{tikzcd} \rightsquigarrow \begin{tikzcd} \mathrm{In}(\mathrm{Hecke}_{G,\le\mu,C}) \arrow[r] & \mathrm{In}(\mathrm{Bun}_{G,C}). \\ \mathrm{In}(\mathrm{Hecke}^{1,*}_{G,\le\mu,C}) \arrow[r] \arrow[u] & \mathrm{In}(\mathrm{Bun}^{1}_{G,C}) \arrow[u] \end{tikzcd}$$

$$\mathrm{In}(\mathrm{Bun}^1_{G,C}) = [\underline{G(F)} /\!/ \underline{G(F)}]$$

$$\begin{tikzcd} \mathcal{E}_1 \arrow[r, dashed] \arrow[d, "g_1"'] \arrow[dr, phantom, "\circlearrowright"] & \mathcal{E}_2 \arrow[d, "g_2"] \\ \mathcal{E}_1 \arrow[r, dashed] & \mathcal{E}_2 \end{tikzcd}$$

g_1 strongly regular ss $\longrightarrow [\underline{G(F)}_{sr} /\!/ \underline{G(F)}] \subset [\underline{G(F)} /\!/ \underline{G(F)}]$

g_1 elliptic sr ss $\longrightarrow [\underline{G(F)}_{ell} /\!/ \underline{G(F)}] \subset [\underline{G(F)}_{sr} /\!/ \underline{G(F)}]$

Let $\begin{tikzcd} \mathcal{E}^1 \arrow[r, dashed, "x"] \arrow[d, "g"'] & \mathcal{E}^b \arrow[d, "g'"] \\ \mathcal{E}^1 \arrow[r, dashed, "x"] & \mathcal{E}^b \end{tikzcd}$ be an object of $\mathrm{In}(\mathrm{Hecke}_{G,\le\mu,C})(C)$ with $(g,g') \in G(F)_{sr} \times G_b(F)_{sr}$.

$$\Downarrow$$

$\pi_2(x) \in \mathrm{Gr}^{1,g}_{G,\le\mu,C}$ (g-fixed pt)

$\pi_1(x) \in \mathrm{Gr}^{b,g'}_{G,\le\mu,C}$.

$$\left(\begin{array}{l} \text{Take } T = \mathrm{Cent}(g,G) \rightsquigarrow \mathrm{Gr}^{1}_{T,\le\mu,C} \hookrightarrow \mathrm{Gr}^{1,g}_{G,\le\mu,C} \\ \qquad \& \ G \text{ sr ss} \Rightarrow \underbrace{\mathrm{Gr}^{1}_{T,\le\mu,C}}_{= X_*(T)_{\le\mu}} = \mathrm{Gr}^{1,g}_{G,\le\mu,C}. \end{array} \right)$$

$\Rightarrow \pi_2(x) \longleftrightarrow \lambda \in X_*(T)_{\le\mu}$, $\pi_1(x) \longleftrightarrow \lambda' \in X_*(T')_{\le -\mu}$.

$T' = \mathrm{Cent}(g', G)$.

<u>Recall</u> for each b, $G_{b,\breve{F}} \hookrightarrow G_{\breve{F}}$.

Prop g & g' are related, i.e. g & g' are conjugate in $G(\overline{\breve{F}})$ (hence in $G(\breve{F})$).

Moreover, $\exists\, y \in G(\breve{F})$ s.t. $g' = ygy^{-1}$. $\mathrm{ad}(y): G_{\breve{F}} \longrightarrow G_{\breve{F}}$

induces an F-rational isom $T_{\breve{F}} \to T'_{\breve{F}}$

s.t. $g \mapsto g'$, $\lambda \mapsto -\lambda'$.

Proof Consider

$$\begin{tikzcd}
\mathcal{E}'|_{X_C \setminus D_e} \arrow[r, "x"] \arrow[d, "g"'] & \mathcal{E}^b|_{X_C \setminus D_e} \arrow[r, phantom, "\simeq"] \arrow[d, "g'"] & \mathcal{E}'|_{X_C \setminus D_e} \arrow[d, "g''"] \\
\mathcal{E}'|_{X_C \setminus D_e} \arrow[r, "x"] & \mathcal{E}^b|_{X_C \setminus D_e} \arrow[r, phantom, "\simeq"] & \mathcal{E}'|_{X_C \setminus D_e}
\end{tikzcd}$$

where g'' = the image of g' along $G_b(F) \longrightarrow G(\breve{F})$.

$\Rightarrow$ g & g' conjugate in $G(B_e)$, $B_e = \mathcal{O}_X(X_C \setminus D_e)$

Let $y \in G(\breve{F})$ s.t. $(\mathrm{ad}(y))(g) = g'$

$\Rightarrow$ $(\mathrm{ad}(y))(T_{\breve{F}}) = T'_{\breve{F}}$.

$\Rightarrow$ $\mathrm{ad}(y): T_{\breve{F}} \longrightarrow T'_{\breve{F}}$, $\breve{F}$-isom.

<u>Claim</u> $\mathrm{ad}(y)$ is F-rational

- Galois action on $T_{\breve{F}}$ is the standard one.
- Galois action on $T'_{\breve{F}} \subseteq G_{\breve{F}}$ is the twisted one.

$\sigma \in \mathrm{Aut}(\breve{F}/F)$ Frob, acting on $T'_{\breve{F}} = \mathrm{int}(b) \circ \sigma$

Let $b_0 := y^{-1} b\, \sigma(y) \in T(\breve{F})$ ($\Leftarrow$ $g \in G(F)$, $g' \in G_b(F)$.)

$\lambda_0 := \mathrm{ad}(y^{-1})(\lambda') \in X_*(T)$.

The element y provides an isom of G-bundles:

$$\begin{tikzcd}
\mathcal{E}^{b_0} \arrow[r, "y"] \arrow[d, dashed] \arrow[dr, phantom, "\circlearrowright"] & \mathcal{E}^b \arrow[d, dashed] \\
\mathcal{E}^b[\lambda_0] \arrow[r, "y"] & \mathcal{E}^b[\lambda']
\end{tikzcd}
\Rightarrow
\begin{tikzcd}
 & \mathcal{E}'[\lambda] \arrow[d, "\cong", "g'"'] & \\
g \circlearrowright \mathcal{E}' \arrow[ur, dashed] \arrow[r, dashed, "x"] \arrow[dr, bend right, "\cong"'] & \mathcal{E}^b \circlearrowleft g' \arrow[d, dashed] & \mathcal{E}^{b_0} \circlearrowleft g \arrow[l, "y^{-1}"'] \arrow[d, dashed] \\
 & \mathcal{E}^b[\lambda'] & \mathcal{E}^{b_0}[\lambda_0] \arrow[l, "y^{-1}"']
\end{tikzcd}$$

$y^{-1}x$ comes from a modification of T-bundles.

$\kappa(\mathcal{E}^{b_0}) = \kappa(\mathcal{E}^1) +$ the class of λ in $\pi_1(T)_\Gamma = X_*(T)$.

$\Rightarrow \lambda = -\lambda_0$ in $X_*(T)$.

<u>Def'n</u> $g \in G(F)_{sr}$, $g' \in G_b(F)_{sr}$ related.

Let $y \in G(\breve{F})$, $g' = ygy^{-1}$, $T = \mathrm{Cent}(g, G)$.

$\Rightarrow b_0 = y^{-1} b y^{\sigma} \in T(\breve{F})$.

Put $\mathrm{inv}[b](g, g') = [b_0] \in B(T)$.

<u>Fact</u> $\forall z \in G(F)$, $z' \in G_b(F)$,

$$\mathrm{inv}[b](\mathrm{ad}(z)(g), \mathrm{ad}(z')(g')) = \mathrm{ad}(z)(\underbrace{\mathrm{inv}[b](g,g')}_{\in B(T)}) \in B(\mathrm{ad}(z)(T)).$$

<u>Def'n</u>

$$\mathrm{Rel}_{b,(\le\mu)} := \left\{ \begin{array}{l} (g, g', \lambda) \in G(F)_{sr} \times G_b(F)_{sr} \times X_*(G) : \\ \quad \text{(i) } g, g' \text{ related,} \\ \quad \text{(ii) } T = \mathrm{Cent}(g, G),\ \lambda \in X_*(T)_{(\le\mu)} \\ \quad \text{(iii) } \kappa(\mathrm{inv}[b](g, g')) = \text{class of } \lambda \end{array} \right\} \Big/ \simeq.$$

where $(g, g', \lambda) \equiv (\mathrm{ad}(z)(g), \mathrm{ad}(z')(g'), \mathrm{ad}(z)(\lambda))$ modulo $\simeq$.

Hence the previous prop $\Rightarrow$

$$\begin{tikzcd} \mathcal{E}^1 \arrow[r, dashed] \arrow[d, "\cong"'] \arrow[dr, dashed] & \mathcal{E}^1[\lambda] \arrow[d, "\cong"] \\ \mathcal{E}^{b_0}(\lambda_0) & \mathcal{E}^{b_0} \arrow[l, dashed] \end{tikzcd}$$

$\Rightarrow b_0 = y^{-1} b y^{\sigma}$, $\mathrm{inv}[b](g, g') = [b_0] \in B(T)$.

$\kappa(b_0) =$ class of λ in $X_*(T)_\Gamma$.

$\Rightarrow (g, g', \lambda) \in \mathrm{Rel}_{b, \le\mu}$.

In particular, get a map

$$|\mathrm{Irr}(\mathrm{Hecke}^{1,b}_{G,\le\mu,c})_{\mathrm{ell}}| \xrightarrow{\sim} \mathrm{Rel}_{b,\mu,\mathrm{ell}}$$

via $\left(\begin{tikzcd} \mathcal{E}^1 \arrow[r, dashed] \arrow[d, "g"'] & \mathcal{E}^b \arrow[d, "g'"] \\ \mathcal{E}^1 \arrow[r, dashed] & \mathcal{E}^b \end{tikzcd}\right) \longmapsto (g, g', \lambda)$.

<u>Thm</u> Assume $[b] \in B(G,\mu)_{bas}$, $g \in G(F)_{ell}$, $g' \in G_b(F)_{ell}$

$\lambda \in X_*(T)_{\le\mu}$, $T = \mathrm{Cent}(g, G)$.

Then $\mathcal{E}^1[\lambda] \cong \mathcal{E}^b$, and hence

$$\mathrm{Gr}^{1,g}_{G,\le\mu} \subseteq \mathrm{Gr}^{1,adm}_{G,\le\mu}.$$

Similarly, $\mathrm{Gr}^{b,g'}_{G,\le\mu} \subseteq \mathrm{Gr}^{b,adm}_{G,\le\mu}$.

<u>Claim</u> $\mathcal{E}^1[\lambda]$ is semisimple.

pf. $\mathcal{E}^1[\lambda]$ comes from a T-bundle.

So only need to check
$$\begin{tikzcd} B(T) \arrow[r] \arrow[dr] & B(G) \\ & B(G)_{bas} \arrow[u] \end{tikzcd}$$

Let $b_0 \in B(T)$, $\mathcal{E}^{b_0} \longrightarrow \nu_{b_0} : \mathbb{D} \longrightarrow T/F$ Newton vector.
$$\begin{tikzcd} \mathbb{D} \arrow[r] \arrow[d] & T/F \\ Z(G)^{\circ} \arrow[ur] & \end{tikzcd}$$

<u>Cor</u> $b \in B(G,\mu)$. $|\mathrm{In}(\mathrm{Hecke}^{1,b}_{G,\le\mu})_{ell}| \simeq \mathrm{Rel}_{b,\mu,ell}$.

<u>§3 Transfer</u>

- b basic, $\mathrm{Rel}_b \subseteq (G(F)_{sr} \times G_b(F)_{sr} \times X_*(G))/G(F) \times G_b(F)$.

$$\begin{tikzcd} & \mathrm{Rel}_b \arrow[dl] \arrow[dr] & \\ G(F)_{sr} /\!/ G(F) & & G_b(F)_{sr} /\!/ G_b(F) \end{tikzcd}$$

Define $T^{G\to G_b}_{b,\mu} : \mathcal{C}_c(G(F)_{sr} /\!/ G(F), \Lambda) \longrightarrow \mathcal{C}_c(G_b(F)_{sr} /\!/ G_b(F), \Lambda)$.

$$T_{b,\mu}^{G\to G_b}(f)(g) := (-1)^d \sum_{(g,g';\lambda)\in \mathrm{Rel}_b} f(g)\cdot \dim V_\mu[\lambda]. \quad d = \langle 2\mu, 2\rho_G\rangle.$$

Also define $T_{b,\mu}^{G_b\to G}$ similarly. V_μ = Weyl module.

$$C_c(G(F)_{sr}/\!/G(F), \Lambda) \xleftarrow{\sim} C_c(G(F)_{sr}, \Lambda)_{G(F)}$$

$$\phi_G \longleftarrow\!\shortmid \phi$$

$$\begin{tikzcd}
\Rightarrow T_{b,\mu}^{G\to G_b} : C_c(G(F)_{sr}, \Lambda)_{G(F)} \arrow[r] & C_c(G_b(F)_{sr}, \Lambda)_{G_b(F)} \\
C_c(G(F)_{ell}, \Lambda)_{G(F)} \arrow[u] \arrow[r, "\mathcal{T}_{b,\mu}^{G\to G_b}"'] \arrow[ur, phantom, "\circlearrowleft"] & C_c(G_b(F)_{ell}, \Lambda)_{G_b(F)} \arrow[u]
\end{tikzcd}$$

<u>Def'n</u> $\mathcal{T}_{b,\mu}^{G_b\to G} : \mathrm{Dist}(G_b(F)_{ell}, \Lambda) \xrightarrow{G_b(F)} \mathrm{Dist}(G(F)_{ell}, \Lambda)^{G(F)}$ is the Λ-dual of $\tilde{\mathcal{T}}_{b,\mu}^{G\to G_b}$.

<u>Prop</u> $\Lambda = \bar{\mathbb{Q}}_\ell$, $\mathcal{T}_{b,\mu}^{G_b\to G}$ extends the transfer $T_{b,\mu}^{G_b\to G}$ in [HKW] §3.