Transfer [HKW, §3]

Fix G conn red gp $/ F$, $F/\mathbb{Q}_p$.

$$\begin{array}{l} G \\ \cup\!\mid \\ G_{rs} \quad \text{reg ss locus} \\ \cup\!\mid \\ G_{sr} \quad \text{strongly rs, i.e. } Z_G(g) \text{ is a max torus} \end{array}$$

<u>Lem 3.1.1</u> $\quad G_{rs}(F)/\!/G(F) = \coprod_{\substack{T \text{ max tori} \\ /\text{conj}}} T_{rs}(F)/N(T,G)(F)$

where $T_{rs} := T \cap G_{rs}$, $N(T,G) := N_G(T)$.

(Replacing rs with sr: similar result.)

<u>Then</u> $G_{rs}(F)/\!/G(F)$ locally profinite w.r.t. quotient top.

<u>Proof</u> Fix T.

$$\begin{array}{ccc} & & \text{finite Gal cohom} \\ \{\text{max tori } T' \subseteq G\}/\text{conj} & \xleftrightarrow{1:1} & \operatorname{Ker}(H^1(F, N(T,G)) \to H^1(F,G)). \\ (T' = xTx^{-1} : x \in G(F)) & \longleftrightarrow & (\tau \mapsto x^{-1}\tau(x),\ \tau \in \Gamma := \mathrm{Gal}_F) \end{array}$$

$\psi : G_{rs}(F)/\!/G(F) \to \{\text{max tori } T' \subseteq G\}/\text{conj}$

Claim: ψ is locally const

b/c $\quad G(F) \times T'_{rs}(F) \longrightarrow G_{rs}(F)$ is open submersive

$\quad (g, t') \longmapsto g t' g^{-1}$ by computing Jacobians

$(\text{Fiber of } \psi \text{ over } T'/\text{conj}) = \underbrace{T'_{rs}(F)}_{\text{loc profin}} / \underbrace{\mathrm{Weyl}(T,G)}_{\text{fin}} \twoheadleftarrow N(T,G)(F)$

$\square$

<u>Lem 3.2.1</u> Let $b \in G(\breve{F}) \rightsquigarrow G_b$, $[b] \in B(G)_{bas}$

s.t. $G_b(F) = \{g \in G(\breve{F}) \mid b\sigma(g)b^{-1} = g\}$

Then $g \in G_{sr}(F) \sim g' \in G_{b,sr}(F)$ conj $/ \overline{\breve{F}} \iff$ conj $/ \breve{F}$.

<u>Proof</u> Diff between $\breve{F}$-conj & $\overline{\breve{F}}$-conj is measured by

$$\operatorname{Ker}\left(H^1(\breve{F}, T) \to H^1(\breve{F}, G)\right), \quad T \subset G \text{ max torus}$$
$$\overset{\|}{\{0\}} \text{ (Steinberg)} \qquad \square$$

Suppose $g' = y g y^{-1}$, $y \in G(\breve{F})$.

Define $b_0 := y^{-1} b\, \sigma(y) \in T(\breve{F})$, $T := Z_G(g)$.

Ambiguity: $y \rightsquigarrow yt$ with $t \in T(\breve{F})$

$b_0 \rightsquigarrow b_0 \sigma(t) t^{-1}$.

<u>Def 3.2.2</u> $\mathrm{inv}[b](g, g') := [b_0] \in B(T)$.

<u>Fact</u> (1) $\mathrm{inv}[b]((\mathrm{Ad}\, z)(g), g') = (\mathrm{Ad}\, z)(\mathrm{inv}[b](g, g'))$, $z \in G(F)$.

(2) $\mathrm{inv}[b](g, (\mathrm{Ad}\, w)(g')) = \mathrm{inv}[b](g, g')$, $w \in G_b(F)$.

(3)
$$\begin{array}{ccccc} B(T) & \xrightarrow{T \subset G} & B(G) & \xrightarrow{\kappa} & \pi_1(G)_\Gamma := X_*(T)/\langle \text{coroots} \rangle \\ \text{\rotatebox{90}{$\in$}} & & & & \\ \mathrm{inv}[b](g, g') & & \longmapsto & & \kappa(b) \quad \circlearrowright \Gamma. \end{array}$$

<u>Def 3.2.4</u> Let

$$\mathrm{Rel}_b := \left. \left\{ (g, g', \lambda) \;\middle|\; \begin{array}{l} g \in G_{sr}(F),\ g' \in G_{b,sr}(F),\ \lambda \in X_*(T), \\ g \,\&\, g' \text{ related s.t. } \kappa_T(\mathrm{inv}[b](g, g')) \\ \qquad\qquad\qquad \overset{\|}{\text{image of } \lambda \text{ in } X_*(T)_\Gamma} \end{array} \right\} \right/$$

$(g, g', \lambda) \sim (\mathrm{Ad}(z)(g), \mathrm{Ad}(w) g', \mathrm{Ad}(z) \lambda)$.

equipped with quotient top on $G(F) \times G_b(F) \times X_*(T) \leftarrow$ disc top.

$\rightsquigarrow$ Define
$$\begin{array}{ccc} & \mathrm{Rel}_b & \\ \swarrow & & \searrow \\ G_{sr}(F) /\!\!/ G(F) & & G_{b,sr}(F) /\!\!/ G_b(F). \end{array}$$

<u>Lem 3.2.6</u> Both maps above are homeomorphic locally on the source.

<u>Proof</u> $G_{sr}(F) /\!\!/ G(F) = \coprod_{T/\mathrm{conj}} T_{sr}(F)/\mathrm{Weyl}$

$$\begin{tikzcd} \mathrm{Rel}_b \arrow[r, hook] \arrow[d] & \coprod_{T/\mathrm{conj}} \left(T_{sr}(F)/\mathrm{Weyl} \times X_*(T)\right). \arrow[dl, bend left, "pr_1"] \\ G_{sr}(F)/\!/G(F) & \end{tikzcd}$$

$\Rightarrow \mathrm{Rel}_b \to G_{sr}(F)/\!/G(F)$ local homeo.

(the right arrow : similar). $\square$

Henceforth, $b \in G(\breve{F})$, $[b] \in B(G)_{bas}$

(up to pinning) $(\hat{G}, \hat{B}, \hat{T})$: L-gp data for G.

$[\mu]$: Γ-stable conj class of $\mu : \mathbb{G}_m \to G_{\bar{F}}$.

May assume $\mathrm{im}(\mu) \subset T_{\bar{F}}$, $T \subset G$ chosen.

$\hat{\mu} : \hat{T} \to \mathbb{G}_m$ $\hat{B}$-dominant.

r_μ : fin dim'l $\hat{G}$-irrep with $\hat{B}$-highest wt $= \hat{\mu}$

$\forall \lambda \in X_*(T) \longleftrightarrow \hat{\lambda} \in X^*(\hat{T})$, $r_\mu[\lambda]$: $\hat{\lambda}$-weight subspace.

$\forall \Lambda$ comm ring, $p \in \Lambda^\times$,

X top space $\rightsquigarrow C(X, \Lambda) := \{\text{cont } X \to \Lambda\}$.

<u>Def 3.2.7</u>

$$\begin{tikzcd} f \arrow[r, mapsto] & \left(g' \mapsto \sum_{(g,g',\lambda) \in \mathrm{Rel}_b} f(g) (-1)^{\langle \hat{\mu}, 2\rho_G \rangle} \dim r_\mu[\lambda]\right) \\ C(G(F)_{sr}/\!/G(F), \Lambda) \arrow[r, shift left, "T_{b,\mu}^{G \to G_b}"] & C(G_b(F)_{sr}/\!/G_b(F), \Lambda) \arrow[l, shift left, "T_{b,\mu}^{G_b \to G}"] \\ \left(g \mapsto (-1)^{\langle \hat{\mu}, 2\rho_G \rangle} \sum_{(g,g',\lambda) \in \mathrm{Rel}_b} f'(g') \dim r_\mu[\lambda]\right) & f' \arrow[l, mapsto] \end{tikzcd}$$

Ambiguity : $\hat{\mu} \in X^*(\hat{T})$ $\hat{B}$-dominant.

$\rho_G = \frac{1}{2} \sum \text{coroots} > 0$ w.r.t. $(\hat{B}, \hat{T})$

$$\begin{tikzcd} \lambda \in X_*(T) \arrow[rr, "\sim"] \arrow[dr, "\sim"'] & & X^*(\hat{T}) \\ & X_*(T_{std}) \arrow[ur, "\sim"'] & \end{tikzcd}$$

unique up to Weyl

$\exists!$ basic element in $B(G,\mu) \subset B(G)$.

Assume $[b] \in B(G,\mu)_{\mathrm{bas}}$.

<u>Lem 3.2.8</u> $T_{b,\mu}^{G\to G_b} = 0$ unless $[b] \in B(G,\mu)$

<u>Sketch pf</u> If $T_{b,\mu}^{G\to G_b} \neq 0$ then $\exists [g, g', \lambda] \in \mathrm{Rel}_b$

s.t. $\hat{\mu}|_{Z(\hat{G})} = \hat{\lambda}|_{Z(\hat{G})} \Rightarrow$ Same image in $X^*(Z(\hat{G})^\Gamma)$

$\lambda \mapsto [b]$ under $X_*(T) \longrightarrow B(T) \longrightarrow B(G)$. $\square$

<u>Thm 3.2.9</u> Λ alg closed, $\Lambda \simeq \mathbb{C}$,

$\phi : W_F \times SL(2,\Lambda) \longrightarrow {}^LG(\Lambda)$ discrete. $\rho \in \Pi_\phi(G_b)$.

Let $\Theta_\rho \in C(G_{b,\mathrm{sr}}(F) /\!/ G_b(F), \Lambda)$ H-C character.

"“trace of ρ"

Same for Θ_π, $\forall \pi \in \Pi_\phi(G)$,

<u>Then</u> $\forall g \in G_{\mathrm{sr}}(F)$ transfering to $G_b(F)$,

$$(T_{b,\mu}^{G_b\to G}\Theta_\rho)(g) = \sum_{\pi \in \Pi_\phi(G)} \dim \mathrm{Hom}_{S_\phi}(\delta_{\pi,\rho}, r_\mu)\,\Theta_\pi(g).$$

where $S_\phi := Z_{\hat{G}}(\mathrm{im}\,\phi)$ (finite mod $Z(\hat{G})^\Gamma$).

<u>E.g.</u> $G = GL_2$, $\mu : \mathbb{G}_m \longrightarrow G$, $r_\mu = \mathrm{Std}$, $\pi_1(G) = \mathbb{Z} \circlearrowleft \Gamma$ trivially.

$$x \longmapsto \begin{pmatrix} x & \\ & 1 \end{pmatrix}$$

$G_b = D^\times$, D quat div alg $/F$. $\Pi_\phi(G)$, $\Pi_\phi(G_b)$: singletons.

We have $S_\phi = Z(\hat{G}) = \Lambda^\times$, $\delta_{\pi,\rho} = \mathrm{id} : \Lambda^\times \longrightarrow \Lambda^\times$.

$$\dim \mathrm{Hom}_{S_\phi}(\delta_{\pi,\rho}, \underset{\substack{\uparrow \\ \mathrm{Std}}}{r_\mu}) = 2$$

- RHS of Thm $= 2\Theta_\pi(g)$.
- LHS of Thm: for $[E:F] = 2$,

$\forall g$, $S := Z_G(g) \simeq \mathrm{Res}_{E/F}\mathbb{G}_{m,E}$.

$X_*(S) = \mathbb{Z}[\Gamma_{E/F}] \circlearrowleft \Gamma$ (ell part).

$\pi_1(S)_\Gamma \xrightarrow{\sim} \pi_1(G)_\Gamma \simeq \mathbb{Z}$.

Fixing g & g', for $(g, g', \lambda) \in$ Relb,

wt space $\mathrm{Std}[\lambda] \neq 0 \Rightarrow$ only 2 choices of λ: $\lambda, w\lambda$ for $w \in \mathrm{Weyl}(S, G)$.

$\Rightarrow$ LHS $= 2\Theta_p(g)$.

Up to $(-1)^{\langle \hat{\mu}, \rho_G \rangle}$, we get the classical JLC.

$\forall n \geq 1$, let $Z_n := \{ z \in Z(G) \mid z^n \in Z(G_{der}) \}$. $G_n := G/Z_n$,

$C_1 := Z(G)/Z(G_{der})$, $C_n := C_1/C_1[n]$.

Then $G_n \simeq G_{ad} \times C_n$ b/c $G = G_{der} \cdot Z(G)$

note $n \mid m \Rightarrow C_n \twoheadrightarrow C_m$ $(C[n] \subset C[m])$.

Consider $\hat{\bar{G}} := \varprojlim_n \hat{G}_n$, $n \mid m \Rightarrow \hat{G}_n \twoheadleftarrow \hat{G}_m$, $G_n \twoheadrightarrow G_m$.

$\quad\quad \parallel$ $\hat{G}_{sc} \times \hat{C}_\infty = \hat{G}_{sc} \times \varprojlim_n \hat{C}_n$.

Have $\hat{G}_{sc} \twoheadrightarrow \hat{G}_{der} \hookrightarrow \hat{G}$, $\hat{\bar{G}} \longrightarrow \hat{G}$

$a \longmapsto a_{der}$ $\quad (a, (b_n)_n) \longmapsto a_{der} b_1$.

Define $Z(\hat{\bar{G}})^+ \subset S_\phi^+ \subset \hat{\bar{G}}$ as the preimages of $Z(\hat{G}) \subset S_\phi \subset \hat{G}$ $(\twoheadleftarrow \hat{\bar{G}})$.

<u>Refined LLC</u> Fix qs inner twist $\psi: G^*_{\bar{F}} \xrightarrow{\sim} G_{\bar{F}}$ (Whittaker data $\mathfrak{w}$ of G^*).

ϕ: disc L-para of G.

$\rightsquigarrow \Pi_\phi(G) \xrightarrow{\sim} \mathrm{Irr}(\pi_0(S_\phi^+), \lambda)$

$\pi \longmapsto \tau_{z, \mathfrak{w}, \pi}$.

(1) $\lambda: \pi_0(Z(\hat{\bar{G}})^+) \longrightarrow \Lambda^\times \simeq \mathbb{C}^\times$.

$[z] \in H^1(u \to W_F, Z(G^*) \to G^*)$ by Kaletha.

(2) For $[z]$, $\exists\, \mathfrak{z} \in Z^1(u \to W_F, Z(G^*) \to G^*)$

s.t. $z \mapsto \xi \in Z^1(F, G^*_{ad})$ $\rightsquigarrow$ ψ.

Call (ψ, z) a "rigid inner twist".

Given $[b] \in B(G)_{bas}$, we obtain

$$\Pi_\phi(G_b) \simeq \mathrm{Irr}(\pi_0(S_\phi^+), \lambda + \lambda_{zb})$$

- (ψ, z) rigid inner twist of G
- $(\psi, \psi^{-1}(z) z_b)$ — $G_b \simeq G_{zb}$.
- $\delta_{\pi,\rho} :=$ image of $\check{\tau}_{z,w,\pi} \otimes \tau_{z,w,\rho} \in \mathrm{Rep}(\pi_0(S_\phi^+), \lambda_{zb})$.
- $\lambda_b = \kappa(b) \in X^*(Z(\hat{G})^\Gamma)$.

$$\mathrm{Rep}(\pi_0(S_\phi^+), \lambda_{zb}) \xrightarrow[\text{Kaletha}]{S} \mathrm{Rep}(S_\phi, \lambda_b).$$

(2.3.2) $S_\phi^+ \longrightarrow S_\phi$

<u>Lem 2.3.3</u> $\delta_{\pi,\rho}$ indep of w & z.

<u>Proof of the main thm</u>

Let $g' \in G_{b,sr}(F)$. Endoscopic character relation for G_b

$\Uparrow$ conj : by Kaletha.

$s \in S_\phi$ ss, $\dot{s} \in S_\phi^+$ lift of s

$\rightsquigarrow$ refined endo datum $(H, \mathcal{H}, \dot{s}, \eta)$.

$$\Rightarrow \quad e(G_b) \sum_{\rho' \in \Pi_\phi(G_b)} \mathrm{tr}(\tau_{z,w,\rho'})(\dot{s})\, \Theta_{\rho'}(g')$$

$$= \sum_{h_1 \in H_{1,sr}(F)/\mathrm{st\ conj}} \Delta(h_1, g')\, \underbrace{S\Theta_{\phi^s}(h_1)}_{\text{st char}}$$

$$= \sum_{h_1 \in H_{1,sr}(F)/\mathrm{st\ conj}} \Delta(h_1, g')\, \langle \underbrace{\mathrm{inv}[b](g, g')}_{\substack{\cap \\ X_*(T)_\Gamma}}, \underbrace{s^\natural_{h,g}}_{\substack{\cap \\ \hat{T}^\Gamma}} \rangle\, S\Theta_{\phi^s}(h_1).$$

$\lambda \in X_*(T) \longrightarrow X_*(T)_\Gamma$

$$\begin{tikzcd} W_F \times SL_2 \arrow[r, "\phi^s"] \arrow[dr, "\phi"'] \arrow[d] & {}^L H_1 \arrow[d, dashed] \\ \boxed{\hat{G}} & \hat{G} \end{tikzcd}$$

Vary g'. fix $g \in G_{sr}(F)$, and multiply $\dim r_\mu[\lambda]$.

Taking sum over $(g, g', \lambda) \in \mathrm{Rel}_b$:

$$e(G_b) \sum_{(g',\lambda)} \sum_{\rho' \in \Pi_\phi(G_b)} \mathrm{tr}\, \tau_{\mathfrak{z},w,\rho'}(\dot{s})\, \Theta_{\rho'}(g') \cdot \dim r_\mu[\lambda]$$

$$= \sum_{h_1} \Delta(h_1, g)\, S\Theta_{\phi^s}(h_1) \sum_{(g',\lambda)} \lambda(s^\natural_{h,g}) \dim r_\mu[\lambda]$$

$$\overset{(*)}{=} \sum_{h_1} \Delta(h_1, g)\, S\Theta_{\phi^s}(h_1)\, \mathrm{tr}(r_\mu(s^\natural_{h,g}))$$

$$\overset{(**)}{=} \mathrm{tr}(r_\mu(s^\natural)) \sum_{h_1} \Delta(h_1, g)\, S\Theta_{\phi^s}(h_1)$$

b/c $s^\natural_{h,g}$ has image in $\widehat{G} \overset{\mathrm{conj}}{\sim} s^\natural :=$ image of $\dot{s} \in S_\phi^+$ in S_ϕ under (2.3.2).

Multiply by $\mathrm{tr}\, \check{\tau}_{\mathfrak{z},w,\rho}(\dot{s}) \Rightarrow Z(\hat{\widehat{G}})^+$-invariant

$\Rightarrow$ descends to $\bar{S}_\phi := S_\phi^+ / Z(\widehat{G})^+$.

Take $\frac{1}{|\bar{S}_\phi|} \sum_{s \in \bar{S}_\phi}$ & use endo char relation for G

$$|\bar{S}_\phi|^{-1} e(G_b) \sum_{\bar{s} \in \bar{S}_\phi} \sum_{(g,\lambda)} \sum_{\rho' \in \Pi_\phi(G_b)} \mathrm{tr}\, \check{\tau}_{\mathfrak{z},w,\rho}(\dot{s}) \cdot \mathrm{tr}\, \tau_{\mathfrak{z},w,\rho'}(\dot{s}) \cdot \Theta_{\rho'}(g') \cdot \dim r_\mu[\lambda]$$

$$= |\bar{S}_\phi|^{-1} e(G_b) \sum_{\bar{s} \in \bar{S}_\phi} \mathrm{tr}\, r_\mu(s^\natural) \sum_{\pi \in \Pi_\phi(G)} \mathrm{tr}\, \check{\tau}_{\mathfrak{z},w,\rho}(\dot{s}) \cdot \mathrm{tr}\, \tau_{\mathfrak{z},w,\rho'}(\dot{s}) \cdot \Theta_\pi(g).$$

& $\mathrm{LHS} = e(G_b) \sum_{(g',\lambda)} \Theta_\rho(g') \cdot \dim r_\mu[\lambda] = e(G_b) \left(T_{b,\mu}^{G_b \to G} \Theta_\rho \right)(g)$ by Fourier inversion

$$\mathrm{RHS} = e(G)\, |\bar{S}_\phi|^{-1} \sum_{\bar{s}} \mathrm{tr}\, r_\mu(s^\natural)\, \mathrm{tr}\, \tilde{\delta}_{\pi,\rho}(s^\natural)$$

$$= e(G) \cdot \dim \mathrm{Hom}_{S_\phi}(\delta_{\pi,\rho}, r_\mu|_{S_\phi}).$$

<u>Remains to prove</u>: $e(G)\, e(G_b) = (-1)^{2\langle \rho_G, \mu \rangle}$.

$b \in B(G,\mu)_{\mathrm{bas}} \overset{(A.2.1)}{\Longrightarrow} e(\cdot)$ defined in [Kottwitz 83]. $\square$