COURSEWORK FOR GEOMETRIC REPRESENTATION THEORY I (SPRING 2024)

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This document is about the course *Geometric Representation Theory I* offered by Qiuzhen College, Tsinghua University, during the Spring 2024 semester. The following contains six sheets of homework problems.

All problems are proposed by the lecturer (referring to lecture notes, which are available at `https://windshower.github.io/linchen/teaching/s2024.html`) and are attached with solutions. The TA is responsible for any mistakes in this document.

Contents

Homework 1

Problem 1.1 (Lecture 2, Exercise 2.6)**.** Prove the following.

(1) The map

$$U(\mathfrak{n}^-) \otimes_k U(\mathfrak{b}) \xrightarrow{\mathsf{mult}} U(\mathfrak{g})$$

is an isomorphism between $(U(\mathfrak{n}^-), U(\mathfrak{b}))$-bimodules.

(2) As an $\mathfrak{n}^-$-module, M_λ is freely generated by v_λ, i.e.,

$$U(\mathfrak{n}^-) \longrightarrow M_\lambda, \quad x \longmapsto x \cdot v_\lambda$$

is an isomorphism.

Solution. (1) It is clear from the construction that mult is a homomorphism of left $U(\mathfrak{n}^-)$-modules and right $U(\mathfrak{b})$-modules. Note that the target $U(\mathfrak{g})$ is regarded as a $(U(\mathfrak{n}^-), U(\mathfrak{b}))$-bimodule via restricting the $U(\mathfrak{g})$-action on itself to $U(\mathfrak{b})$ and $U(\mathfrak{n}^-)$, respectively. It remains to show that mult is an isomorphism of vector spaces by PBW theorem [Lecture 2, Theorem 1.5]. For this, according to the context of [Lecture 2, Corollary 1.6], pick $\{x_1, \dots, x_n\}$ as a basis of $\mathfrak{n}^-$ and $\{y_1, \dots, y_m\}$ as a basis of $\mathfrak{b}$. Then the bases of k-vector spaces $U(\mathfrak{n}^-)$, $U(\mathfrak{b})$, $U(\mathfrak{g})$ are $\{x_1^{k_1} \cdots x_n^{k_n}\}_{k_i \geqslant 0}$, $\{y_1^{l_1} \cdots y_m^{l_m}\}_{l_j \geqslant 0}$, $\{x_1^{k_1} \cdots x_n^{k_n} y_1^{l_1} \cdots y_m^{l_m}\}_{k_i, l_j \geqslant 0}$, respectively; it follows that $U(\mathfrak{n}^-) \otimes_k U(\mathfrak{b})$ has a basis $\{x_1^{k_1} \cdots x_n^{k_n} \otimes y_1^{l_1} \cdots y_m^{l_m}\}_{k_i, l_j \geqslant 0}$. Moreover,

$$\mathsf{mult} \colon x_1^{k_1} \cdots x_n^{k_n} \otimes y_1^{l_1} \cdots y_m^{l_m} \longmapsto x_1^{k_1} \cdots x_n^{k_n} y_1^{l_1} \cdots y_m^{l_m}.$$

This completes the proof that mult is an isomorphism between $(U(\mathfrak{n}^-), U(\mathfrak{b}))$-bimodules.

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(2) Using (1), we have that

$$U(\mathfrak{n}^-) \otimes_k U(\mathfrak{b}) \otimes_{U(\mathfrak{b})} k_\lambda \cong U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} k_\lambda = M_\lambda.$$

Here note that each $x \cdot v_\lambda \in M_\lambda$ can be identified with $x \otimes v_\lambda \in U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} k_\lambda$. On the other hand, the left-hand side is isomorphic to $U(\mathfrak{n}^-) \otimes_k k_\lambda$, which is further isomorphic to $U(\mathfrak{n}^-)$. This completes the proof. □

Problem 1.2 (Lecture 2, Exercise 2.17)**.** In the case $\mathfrak{g} = \mathfrak{sl}_2$, show the Verma module M_l is irreducible unless $l \in \mathbb{Z}_{\geqslant 0}$. In the latter case, show there is a non-split short exact sequence

$$0 \longrightarrow M_{-l-2} \longrightarrow M_l \longrightarrow L_l \longrightarrow 0$$

such that L_l is a finite-dimensional irreducible $\mathfrak{sl}_2$-module with highest weight l.

Solution. Suppose M_l is not irreducible. Then there is a nonzero proper submodule $N \subset M_l$ of highest weight l'; denote by $v_{l'}$ the highest weight vector. In this case l' is also regarded as a weight of M_l. Recall that for $\mathfrak{g} = \mathfrak{sl}_2$, any weight of M_l is of form $l - 2n$ with $n \geqslant 0$. So we may assume $l' = l - 2n$. Since $v_{l'}$ generates N via $\mathfrak{g}$-action, the hypothesis $N \subsetneq M_l$ implies $l' \neq l$ (otherwise $M_l = N$), or equivalently $n > 0$. Let $e, f, g \in \mathfrak{sl}_2$ be the standard generators. The weight of $e \cdot v_{l'}$ is either 0 or $l' + 2$. Since l is the highest weight of M_l, we must be in the former case that $e \cdot v_l = 0$. Thus, we obtain from $[e, f] = h$ and $h \cdot f^j - f^j \cdot h = -2jf^j$ that[1]

$$\begin{aligned} e \cdot f^n \cdot v_l &= \sum_{1 \leqslant i \leqslant n} f^{n-i} \cdot [e, f] \cdot f^{i-1} \cdot v_l + f^n \cdot e \cdot v_l \\ &= \sum_{1 \leqslant i \leqslant n} f^{n-i} \cdot h \cdot f^{i-1} \cdot v_l + f^n \cdot e \cdot v_l \\ &= \sum_{1 \leqslant i \leqslant n} (l - 2(i-1)) \cdot f^{n-1} \cdot v_l \\ &= n(l - (n-1)) \cdot f^{n-1} \cdot v_l. \end{aligned}$$

Since the above is 0 whereas $f^{n-1} \cdot v_l \neq 0$, it implies that $l = n - 1 \geqslant 0$. This shows the irreducibility of M_l for $l < 0$.

If $l = n - 1$ for $n > 0$, the same computation shows that $e \cdot f^n \cdot v_l = e \cdot v_{l-2n} = 0$, with $l - 2n = -l-2$, i.e. the vector $v_{l-2n} \in M_l$ generates a submodule of M_l that is isomorphic to M_{-l-2}. By fixing an isomorphism $L_l \simeq M_l/M_{-l-2}$, we get a quotient module L_l of M_l as well as the desired short exact sequence $0 \to M_{-l-2} \to M_l \to L_l \to 0$. Such L_l is clearly finite-dimensional of highest weight l. If the sequence splits, then there is a nonzero section map $M_l \to M_{-l-2}$, and hence a nonzero vector in M_{-l-2} of weight l, which is impossible. So the sequence is non-split.

It remains to show the irreducibility of L_l. For this, note that whenever $l = n-1$ the highest weight of a proper submodule N of M_l must be $-l-2$, so there is no proper submodule of M_l containing M_{-l-2}. Correspondingly, the quotient L_l must be irreducible. □

Problem 1.3 (Lecture 2, Exercise 3.9)**.** Recall for any $V_1, V_2 \in \mathfrak{g}\text{-}\mathsf{mod}$, the tensor product $V_1 \otimes V_2$ of the underlying vector spaces has a natural $\mathfrak{g}$-module structure defined by $x \cdot (v_1 \otimes v_2) \coloneqq (x \cdot v_1) \otimes v_2 + v_1 \otimes (x \cdot v_2)$.

(1) Prove that if V_1 and V_2 are weight modules, so is $V_1 \otimes V_2$. Determine the weights and weight spaces of $V_1 \otimes V_2$ in terms of those for V_1 and V_2.
(2) Consider the case $\mathfrak{g} = \mathfrak{sl}_2$. Prove that the tensor product of two Verma modules is not contained in $\mathcal{O}$.

Solution. (1) If V_1, V_2 are weight modules, then we can respectively take $V_{1,\lambda} \subset V_1$ and $V_{2,\nu} \subset V_2$ to be λ- and ν-eigenspaces, where $\lambda, \nu \in \mathfrak{t}^*$. For all $t \in \mathfrak{t}$ together with $v_1 \in V_{1,\lambda}$ and $v_2 \in V_{2,\nu}$, we have $t \cdot (v_1 \otimes v_2) = (t \cdot v_1) \otimes v_2 + v_1 \otimes (t \cdot v_2) = \lambda(t) v_1 \otimes v_2 + \nu(t) v_1 \otimes v_2 = (\lambda(t) + \nu(t)) \cdot v_1 \otimes v_2$. It follows that $v_1 \otimes v_2 \in V_1 \otimes V_2$ is of weight $\lambda + \nu$. This proves that $V_1 \otimes V_2$ is a weight module; each weight space of $V_1 \otimes V_2$ is of form $\bigoplus_{\lambda+\nu=\mu} V_{1,\lambda} \otimes V_{2,\nu}$ for some fixed $\mu \in \mathfrak{t}^*$.

[1]There is a typo in the proof of [Gai05, Proposition 1.9], c.f. the third line of the computation.

(2) Let M_λ, M_ν be two Verma modules. We prove $M_\lambda \otimes M_\nu \notin \mathcal{O}$ by showing that it is not finitely generated. Suppose for the sake of contradiction that $M_\lambda \otimes M_\nu$ is generated by $m_1, \dots, m_n$ for some $n \in \mathbb{Z}$. Indeed, $M_\lambda \otimes M_\nu$ is isomorphic to a quotient module of an extension of finitely many Verma modules. When $\mathfrak{g} = \mathfrak{sl}_2$, the weight spaces of each of these Verma modules are all 1-dimensional. (Note that this is not true for general $\mathfrak{g}$.) After taking the quotient towards $M_\lambda \otimes M_\nu$, it follows that the dimension of any weight space is at most n. Also, since $\mathfrak{g} = \mathfrak{sl}_2$, there turns out to be a weight space of $M_\lambda \otimes M_\nu$ of weight $\lambda + \nu - 2n$ and dimension $n+1$. This is impossible, and thus $M_\lambda \otimes M_\nu$ cannot be finitely generated. □

Problem 1.4 (Lecture 3, Exercise 5.6)**.**

(1) Find all maps between k-schemes $\mathbb{A}^1 \to \mathbb{A}^1 \backslash 0$.
(2) Find all 1-dimensional representations of the additive group $\mathbb{G}_\mathrm{a}$.
(3) Find all maps between k-schemes $\mathbb{A}^1 \backslash 0 \to \mathbb{A}^1 \backslash 0$.
(4) Find all 1-dimensional representations of the multiplicative group $\mathbb{G}_\mathrm{m}$.

Solution. (1) It suffices to recognize all homomorphisms $k[x]_x \to k[x]$ between k-algebras; here $k[x]_x$ is the localization of $k[x]$ at the point with coordinate x. Since x is invertible in $k[x]_x$, its image in $k[x]$ must be invertible as well, which implies that the image of x must be an element of $k^\times$. It follows that all $k[x]_x \to k[x]$ (with $1 \mapsto 1$ and $x \mapsto t \in k^\times$) are parameterized by $k^\times$. Therefore, all maps between k-schemes $\mathbb{A}^1 \to \mathbb{A}^1 \backslash 0$ are exactly parametrized by $k^\times$, which are constant maps sending all points on $\mathbb{A}^1$ to some point on $\mathbb{A}^1 \backslash 0$.

(2) Any 1-dimensional representation of $\mathbb{G}_\mathrm{a}$ is given by the map $\mathbb{G}_\mathrm{a} \to \mathbb{G}_\mathrm{m}$ of group schemes over some (algebraically closed) field, say k. At the level of k-schemes, the data of $\mathbb{G}_\mathrm{a} \to \mathbb{G}_\mathrm{m}$ can be specialized to the data of $\mathbb{A}^1 \to \mathbb{A}^1 \backslash 0$. But the result of (1) forces $\mathbb{G}_\mathrm{a} \to \mathbb{G}_\mathrm{m}$ to be a constant map, which can only be the trivial representation of $\mathbb{G}_\mathrm{a}$. In other words, the 1-dimensional representation of $\mathbb{G}_\mathrm{a}$ can only be trivial.

(3) As in (1), we aim to figure out all homomorphisms $k[x]_x \to k[x]_x$ between k-algebras. It suffices to determine the image of x. Note that all invertible elements of $k[x]_x$ are of form tx^n with $t \in k^\times$ and $n \in \mathbb{Z}$. We thus conclude that each map $\mathbb{A}^1 \backslash 0 \to \mathbb{A}^1 \backslash 0$ is parametrized by some $tx^n \in k^\times x^\mathbb{Z}$, sending $P \in \mathbb{A}^1 \backslash 0$ to tP^n.

(4) As in (2), consider the map $\mathbb{G}_\mathrm{m} \to \mathbb{G}_\mathrm{m}$ of group schemes over k. Note that this is determined by $\mathbb{A}^1 \backslash 0 \to \mathbb{A}^1 \backslash 0$, which must be of form $P \mapsto tP^n$ by (3). Moreover, as a group homomorphism, we must have $\mathbb{1} \mapsto \mathbb{1}$ where $\mathbb{1}$ is the identity element of $\mathbb{G}_\mathrm{m}$; it hence implies $t = 1 \in k^\times$. Therefore, the desired 1-dimensional representation of $\mathbb{G}_\mathrm{m}$ must be $\mathbb{G}_\mathrm{m} \to \mathbb{G}_\mathrm{m},\ g \mapsto g^k$ for some $k \in \mathbb{Z}$. □

Problem 1.5 (Lecture 3, Exercise 6.6)**.** Let G be any semisimple algebraic group with Lie algebra $\mathfrak{g}$. Prove any Verma module of $\mathfrak{g}$ is not G-integrable.

Solution. Suppose M_λ is a G-integrable Verma module of $\mathfrak{g}$, i.e. M_λ is a $\mathfrak{g}$-module coming from a representation of G through the functor $\mathsf{Rep}(G) \to \mathfrak{g}\text{-}\mathsf{mod}$. By [Lecture 3, Proposition 5.7], if this is the case, then the G-action on M_λ is locally finite, i.e. M_λ is a union of finite-dimensional subrepresentations.

Let v_λ be the highest weight vector in M_λ. Note that v_λ generates M_λ through the orbit of $\mathfrak{g}$-action. Since M_λ is locally finite, there exists a finite-dimensional $\mathfrak{g}$-submodule containing v_λ that is also a G-invariant subspace; it must equal to M_λ for the prescribed reason. In particular, in this case M_λ is finite-dimensional, which is a contradiction. □

Homework 2

Problem 2.1 (Lecture 4, Exercise 2.2)**.** Let $\mathfrak{g} = \mathfrak{sl}_2$ and e, h, f be the standard basis. Consider
$$\Omega := ef + fe + \frac{1}{2}h^2 \in U(\mathfrak{sl}_2).$$
Calculate its image in $\mathrm{Sym}(\mathfrak{t}) = k[h]$ under the k-linear surjection $U(\mathfrak{g}) \twoheadrightarrow \mathrm{Sym}(\mathfrak{t})$ (see equation (2.1) of Lecture 4).

Solution. Using PBW theorem, the prescribed map can be reinterpreted as
$$U(\mathfrak{g}) \xrightarrow{\sim} U(\mathfrak{n}^-) \otimes_k U(\mathfrak{t}) \otimes_k U(\mathfrak{n}) \longrightarrow U(\mathfrak{t}) \xrightarrow{\sim} \mathrm{Sym}(\mathfrak{t}) = k[h].$$
Recall that the elements of the standard PBW basis of $U(\mathfrak{n}^-) \otimes_k U(\mathfrak{t}) \otimes_k U(\mathfrak{n})$ are of form $f^a \otimes h^b \otimes e^c$ with $a, b, c \in \mathbb{Z}_{\geqslant 0}$, which is the image of $f^a h^b e^c \in U(\mathfrak{g})$ along the first isomorphism above. Motivated by this, we write
$$\Omega = ef + fe + \frac{1}{2}h^2 = [e, f] + 2fe + \frac{1}{2}h^2 = 2fe + h + \frac{1}{2}h^2.$$
On the other hand, note that $f \otimes 1 \otimes e$ and $1 \otimes h \otimes 1$ are respectively sent to 0 and h in $\mathrm{Sym}(\mathfrak{t})$. Thus,
$$\begin{aligned} U(\mathfrak{sl}_2) &\longrightarrow \mathrm{Sym}(\mathfrak{t}) = k[h] \\ ef &\longmapsto 0 \\ h &\longmapsto h. \end{aligned}$$
It follows that the image of Ω is $h + \frac{1}{2}h^2$. $\square$

Problem 2.2 (Lecture 4, Exercise 2.7)**.** Let $\kappa\colon \mathfrak{g} \times \mathfrak{g} \to k$ be any nondegenerate symmetric invariant bilinear form on $\mathfrak{g}$. For any basis $x_1, \dots, x_n$ of $\mathfrak{g}$ and its dual basis $x_1^*, \dots, x_n^*$ with respect to the form κ[2], consider the Casimir element
$$\Omega_\kappa = \sum_{i=1}^{n} x_i \cdot x_i^* \in U(\mathfrak{g}).$$

(1) Prove: the Casimir element Ω_κ does not depend on the choice of the basis, and is contained in the center $Z(\mathfrak{g})$.
(2) For $\mathfrak{g} = \mathfrak{sl}_2$, $\kappa = \mathsf{Kil}$, and the canonical basis e, h, f, find Ω_{Kil} and prove it is not contained in $\mathrm{Sym}(\mathfrak{t}) \subset U(\mathfrak{g})$.

Solution. (1) Note that the nondegenerate symmetric pairing κ is specialized from the map $\mathfrak{g} \otimes \mathfrak{g} \to k$, and it uniquely corresponds to an isomorphism $\phi\colon \mathfrak{g} \to \mathfrak{g}^*$, whose inverse is written as $\phi^{-1}\colon \mathfrak{g}^* \to \mathfrak{g}$. Since ϕ^{-1} is again an isomorphism, there is a unique nondegenerate symmetric pairing $\kappa^\vee\colon k \to \mathfrak{g} \otimes \mathfrak{g}$ determined by ϕ^{-1}. Consider its composite with the natural embedding map $\mathfrak{g} \otimes \mathfrak{g} \to U(\mathfrak{g})$; we claim that
$$\begin{aligned} k \xrightarrow{\kappa^\vee} \mathfrak{g} \otimes \mathfrak{g} &\longrightarrow U(\mathfrak{g}) \\ 1 &\longmapsto \Omega_\kappa, \end{aligned}$$
i.e. the Casimir element is the image of $1 \in k$. Indeed, as $x_1, \dots, x_n$ is a basis of $\mathfrak{g}$, its dual basis of $\mathfrak{g}^*$ with respect to κ is given by $x_1^*, \dots, x_n^*$ with $x_i^* = \phi(x_i)$. So the image of $1 \in k$ in $U(\mathfrak{g})$ is given by that of $(\sum_i x_i) \otimes \left(\sum_i \phi^{-1}(x_i^*)\right)$. With the isomorphism ϕ^{-1} we are able to identify $\phi^{-1}(x_i^*)$ with x_i^* by abuse of notation, and hence the image is the same as $\sum_i x_i \otimes x_i^*$, which further maps to $\sum_i x_i \cdot x_i^* = \Omega_\kappa \in U(\mathfrak{g})$. This proves the claim. To conclude, we see Ω_κ is only determined by $1 \in k$ and the choice of $\kappa^\vee$, so it is independent of the choice of $x_1, \dots, x_n$.

We then check $\Omega_\kappa \in Z(\mathfrak{g})$. For any $V \in \mathfrak{g}\text{-}\mathsf{mod}$, there is a $\mathfrak{g}$-mod structure on V^* given by $(x \cdot f)(v) = f(-x \cdot v)$[3] for $f \in V^*$ and $x \in \mathfrak{g}$. With this $\mathfrak{g}$-action on $V = \mathfrak{g}$ and $V^* = \mathfrak{g}^*$, all prescribed maps $\phi, \phi^{-1}, \kappa, \kappa^\vee$ are $\mathfrak{g}$-linear. It follows that the preimage of Ω_κ in $\mathfrak{g} \otimes \mathfrak{g}$ is already $\mathfrak{g}$-invariant with

[2]By definition, this means $\kappa(x_i, x_j^*)_{i,j}$ is the unit matrix.

[3]Here the negative sign has various explanation: from the point of view of lie algebra, $\exp(-x) = \exp(x)^{-1}$; from the isomorphism $\mathfrak{gl}(V) = \mathfrak{gl}(V^*)^{\mathrm{op}}$; it is the unique action such that $k \to V \otimes V^*$ and $V \otimes V^* \to k$ are $\mathfrak{g}$-linear, where k is the trivial $\mathfrak{g}$-module.

respect to the diagonal $\mathfrak{g}$-action. By definition, this diagonal action is $x\cdot(u\otimes v)=[x,u]\otimes v+u\otimes[x,v]$, and its image in $U(\mathfrak{g})$ is $xuv-uxv+uxv-uvx=xuv-uvx$. Hence any $\mathfrak{g}$-invariant element in $\mathfrak{g}\otimes\mathfrak{g}$ is sent to $Z(\mathfrak{g})$, which implies $\Omega_\kappa\in Z(\mathfrak{g})$. (Formally, the argument above means the tensor $\mathfrak{g}$-module structure of $T(\mathfrak{g})=\bigoplus_{n\geqslant 0}\mathfrak{g}^{\otimes n}$ is $\mathfrak{g}$-linearly compatible with the adjoint $\mathfrak{g}$-module stucture of $U(\mathfrak{g})$. So the $\mathfrak{g}$-invariant elements of $\mathfrak{g}\otimes\mathfrak{g}$ is mapped to the $\mathfrak{g}$-invariant elements of $U(\mathfrak{g})$, namely $Z(\mathfrak{g})$.)

(2) Using the relations $[e,f]=h$, $[h,e]=2e$, and $[h,f]=-2f$, we compute the matrices of ad_e, ad_f, and ad_h under the ordered basis $\mathtt{B}=\{e,f,h\}$ as

$$[\mathrm{ad}_e]_{\mathtt{B}}=\begin{pmatrix}0&0&-2\\0&0&0\\0&1&0\end{pmatrix},\quad [\mathrm{ad}_f]_{\mathtt{B}}=\begin{pmatrix}0&0&0\\0&0&2\\-1&0&0\end{pmatrix},\quad [\mathrm{ad}_h]_{\mathtt{B}}=\begin{pmatrix}2&0&0\\0&-2&0\\0&0&0\end{pmatrix}.$$

From the definition,

$$\mathsf{Kil}(x,y)=\mathrm{tr}(\mathrm{ad}_x\circ\mathrm{ad}_y)=\mathrm{tr}([\mathrm{ad}_x]_{\mathtt{B}}[\mathrm{ad}_y]_{\mathtt{B}}).$$

So we deduce

$$\mathsf{Kil}(e,f)=\mathsf{Kil}(f,e)=4,\quad \mathsf{Kil}(h,h)=8.$$

Then, with respect to the Killing form, the ordered dual basis of $\mathtt{B}$ is given by

$$\mathtt{B}^*=\left\{e^*=\frac{f}{4},\ f^*=\frac{e}{4},\ h^*=\frac{h}{8}\right\}.$$

Therefore,

$$\Omega_{\mathsf{Kil}}=\frac{1}{4}ef+\frac{1}{4}fe+\frac{1}{8}h^2=\frac{1}{2}fe+\frac{1}{4}h+\frac{1}{8}h^2.$$

Here the right-hand side is written as a linear combination of elements in standard PBW basis. Note that $fe\notin k[h]$, and hence $\Omega_{\mathsf{Kil}}\notin\mathrm{Sym}(\mathfrak{t})=k[h]$. □

Alternative Solution. (1) Let $y_1,\dots,y_n$ be another basis of $\mathfrak{g}$ and $y_1^*,\dots,y_n^*$ be its dual basis. Then there are invertible matrices $A=(a_{ij}),B=(b_{ij})\in\mathrm{M}_n(k)$ such that

$$x_i=\sum_{j=1}^n a_{ij}y_j,\quad x_i^*=\sum_{j=1}^n b_{ij}y_j^*.$$

Since κ is a bilinear form, $\kappa(x_i,x_j^*)=\kappa\left(\sum_k a_{ik}y_k,\sum_l b_{jl}y_l^*\right)=\sum_{k,l}a_{ik}b_{jl}\cdot\kappa(y_k,y_l^*)$. On the other hand, by definition we have $\kappa(x_i,x_j^*)=\kappa(y_i,y_j^*)=\delta_{ij}$, where δ_{ij} is the Kronecker symbol. It follows that $\delta_{ij}=\sum_{k,l}a_{ik}b_{jl}\cdot\delta_{kl}=\sum_k a_{ik}b_{jk}$. Changing the indices of the sum, this is equivalent to

$$\delta_{kl}=\sum_{i=1}^n a_{ik}b_{il}.$$

Therefore, to show the independence of the choice of basis, we have

$$\sum_{i=1}^n x_i\cdot x_i^*=\sum_{1\leqslant i,k,l\leqslant n}a_{ik}y_k\cdot b_{il}y_l^*=\sum_{1\leqslant k,l\leqslant n}\delta_{kl}(y_k\cdot y_l^*)=\sum_{k=1}^n y_k\cdot y_k^*.$$

Now it remains to check $\Omega_\kappa\in Z(\mathfrak{g})$, and it suffices to show Ω_κ commutes with any element in $U(\mathfrak{g})$. By definition of the universal enveloping algebra, it further suffices to show Ω_κ commutes with any $y\in\mathfrak{g}$. For this, as $x_1,\dots,x_n$ is a chosen basis of $\mathfrak{g}$, we may assume $y=x_i$ and compute

$$\begin{aligned}x_i\Omega_\kappa-\Omega_\kappa x_i&=\sum_{j=1}^n x_i(x_j\cdot x_j^*)-(x_j\cdot x_j^*)x_i\\&=\sum_{j=1}^n[x_i,x_j]\cdot x_j^*+x_j\cdot[x_i,x_j^*].\end{aligned}$$

For this to be 0, from the assumption on κ we have that $\kappa([x_i,x_j],x_i^*)+\kappa(x_j,[x_i,x_j^*])=0$. So the desired result follows.

(2) The same as in the first solution. □

Problem 2.3 (Lecture 4, Exercise 4.6)**.** Let $\mathfrak{g} = \mathfrak{sl}_2$. Prove $Z(\mathfrak{sl}_2) \simeq k[\Omega_{\mathsf{Kil}}]$ where Ω_{Kil} is the Casimir element. (You can use [Lecture 4, Theorem 4.1] for this exercise.)

Solution. In Problem 2.2(2) we have proved that $\Omega_{\mathsf{Kil}} \notin \mathrm{Sym}(\mathfrak{t})$. However, in this problem Ω_{Kil} denotes (by abuse of notation) the image of $\Omega_{\mathsf{Kil}} \in Z(\mathfrak{sl}_2)$ in Problem 2.2(2) along the map $U(\mathfrak{sl}_2) \twoheadrightarrow \mathrm{Sym}(\mathfrak{t})$ in Problem 2.1 restricted to $Z(\mathfrak{sl}_2)$. Thus, combining the results before we deduce $k[\Omega_{\mathsf{Kil}}] = k[(h + h^2/2)/4] = k[2h + h^2]$. Also recall the Harish-Chandra isomorphism

$$Z(\mathfrak{g}) \simeq \mathrm{Sym}(\mathfrak{t})^{W_\bullet}.$$

It together with the fact that $\mathrm{Sym}(\mathfrak{t}) = k[h]$ for $\mathfrak{g} = \mathfrak{sl}_2$ reduces the proof to showing the set-theoretical equality

$$k[2h + h^2] = k[h]^{W_\bullet}.$$

We know the $W_\bullet$-action on $k[h]$ is given by $h \mapsto -h-2$. It follows for $f(h) \in k[h]$ that if $f(h) \in k[h]^{W_\bullet}$ then $f(h) = f(-h-2)$, and in particular $f(0) = f(-2)$, implying that f is generated by a polynomial in h that simultaneously vanishes at 0 and -2, namely $2h+h^2$; therefore, $k[h]^{W_\bullet} \subset k[2h+h^2]$. Finally, the converse inclusion $k[h]^{W_\bullet} \supset k[2h + h^2]$ is clear because $2h + h^2 = 2(-h-2) + (-h-2)^2$. This completes the proof. □

Problem 2.4 (Lecture 5, Exercise 3.3)**.** Prove: the adjoint $\mathfrak{g}$-action on $U(\mathfrak{g})$, i.e.,

$$\mathfrak{g} \times U(\mathfrak{g}) \longrightarrow U(\mathfrak{g}), \quad (x, u) \longmapsto \mathrm{ad}_x(u) = [x, u],$$

preserves each $\mathsf{F}^{\leqslant n}U(\mathfrak{g})$, and the induced $\mathfrak{g}$-action on $\mathrm{gr}^\bullet(U(\mathfrak{g})) \simeq \mathrm{Sym}(\mathfrak{g})$ is the adjoint action in [Lecture 5, Construction 3.1].

Solution. Since $U(\mathfrak{g})$ is the quotient of $\bigoplus_{n\geqslant 0} \mathfrak{g}^{\otimes n}$, each $\mathsf{F}^{\leqslant n}U(\mathfrak{g})$ contains elements generated by elements in $\mathfrak{g}$ of degree at most n; namely, each $u \in \mathsf{F}^{\leqslant n}U(\mathfrak{g})$ is of form $u = \sum_i a_i \cdot u_{i1} \cdots u_{in_i}$ for some $1 \leqslant n_i \leqslant n$. Without loss of generality we may assume $u = u_1 \cdots u_m \in \mathsf{F}^{\leqslant n}U(\mathfrak{g})$ for some $1 \leqslant m \leqslant n$ with $u_i \in \mathfrak{g}$. Then for $x \in \mathfrak{g}$,

$$\begin{aligned}\mathrm{ad}_x(u) &= x \cdot u - \sum_{i=1}^{m-1} u_1 \cdots u_i \cdot x \cdot u_{i+1} \cdots u_m + \sum_{i=1}^{m-1} u_1 \cdots u_i \cdot x \cdot u_{i+1} \cdots u_m - u \cdot x \\ &= \sum_{i=1}^{m} u_1 \cdots u_{i-1} \cdot [x, u_i] \cdot u_{i+1} \cdots u_m.\end{aligned}$$

Given this, note that for $u_i \in \mathfrak{g}$ we have $[x, u_i] \in \mathfrak{g}$; it then follows that $\mathrm{ad}_x(u) \in \mathsf{F}^{\leqslant n}U(\mathfrak{g})$. Thus the filtration of $U(\mathfrak{g})$ is preserved by the adjoint $\mathfrak{g}$-action.

By PBW theorem the image of $u_1 \cdots u_m \in U(\mathfrak{g})$ in $\mathrm{gr}^\bullet U(\mathfrak{g})$ is $u_1 \otimes \cdots \otimes u_m$. From the formula of $\mathrm{ad}_x(u)$ above, we see under the induced $\mathfrak{g}$-action by $x \in \mathfrak{g}$ on $\mathrm{Sym}(\mathfrak{g})$, we have

$$(x, u_1 \otimes \cdots \otimes u_m) \longmapsto \sum_{i=1}^{m} u_1 \otimes \cdots \otimes u_{i-1} \otimes (x \cdot u_i) \otimes u_{i+1} \otimes \cdots u_m.$$

This is the same as [Lecture 5, Construction 3.1]. □

Problem 2.5 (Lecture 5, Exercise 4.8)**.** Let $\lambda \in P^+$ be a dominant integral weight and $n \geqslant 0$. Prove there exist scalars $c_{\lambda'} \in k$, $\lambda' \prec \lambda$, such that

$$\phi_{\mathrm{cl}}(a_{\lambda,n}) = a_{\lambda,n}|_{\mathfrak{t}} = \frac{1}{\#\mathrm{Stab}_W(\lambda)} b_{\lambda,n} + \sum_{\lambda' \prec \lambda} c_{\lambda'} b_{\lambda',n},$$

where $\mathrm{Stab}_W(\lambda) \subset W$ is the stabilizer of the W-action at λ.

Solution. Let L_λ be the unique finite-dimensional irreducible $\mathfrak{g}$-module with highest weight λ. Denote by $\mathrm{wt}(L_\lambda)$ the set of all weights of L_λ. For any $x \in \mathfrak{t}$, from the definition we have

$$a_{\lambda,n}(x) = \mathrm{tr}(x^n; L_\lambda) = \sum_{\mu \in \mathrm{wt}(L_\lambda)} \dim(L_\lambda)_\mu \cdot \mu^n(x).$$

Consider the group action of W on $\mathrm{wt}(L_\lambda)$[4] with finitely many orbits $W\lambda_0, W\lambda_1, \dots, W\lambda_s$, where $\lambda_0 = \lambda$. Since λ is dominant, each $\lambda' \in \mathrm{wt}(L_\lambda)$ satisfies $\lambda' \prec \lambda$. Thus,

$$\begin{aligned}
\alpha_{\lambda,n}(x) &= \sum_{\mu \in \mathrm{wt}(L_\lambda)} \dim(L_\lambda)_\mu \cdot \mu^n(x) \\
&= \sum_{i=0}^{s} \sum_{\mu \in W\lambda_i} \dim(L_\lambda)_{\lambda_i} \cdot \mu^n(x) \\
&= \sum_{i=0}^{s} \frac{1}{\#\mathrm{Stab}_W(\lambda_i)} \sum_{w \in W} \dim(L_\lambda)_{\lambda_i} \cdot (w\lambda_i)^n(x) \\
&= \frac{1}{\#\mathrm{Stab}_W(\lambda)} b_{\lambda,n}(x) + \sum_{\lambda' \prec \lambda} c_{\lambda'} b_{\lambda',n}(x)
\end{aligned}$$

for some scalars $c_{\lambda'} \in k$. Here $b_{\lambda,n} \coloneqq \sum_{w \in W} w(\lambda^n)$, and the last equality is because $w\lambda_i \prec \lambda$ for $\lambda_i \neq \lambda$ and each $\lambda' \prec \lambda$ has the form $w\lambda_i$ for some w and some $i > 0$. $\square$

[4]The action of W on $\mathrm{wt}(L_\lambda)$ can be realized as follows. As a $\mathfrak{g}$-module, L is G_{sc}-integrable as it is finite-dimensional (in our case $G = G_{\mathrm{sc}}$); this induces the action of $N_T(G)$ on L. On the other hand, corresponding to the action of $\mathfrak{t}$, the action of T on L preserves all weight spaces, and in particular preserves the highest weight space L_λ. It follows that $W \coloneqq N_T(G)/T$ acts on $\mathrm{wt}(L)$, and thus on $\mathrm{wt}(L_\lambda)$.

Homework 3

Problem 3.1 (Lecture 6, Exercise 1.18)**.** Let $\mathfrak{g} = \mathfrak{sl}_n$ and let $\sigma_i \in \mathrm{Fun}(\mathfrak{t})^W$ be the basic symmetric polynomial of degree i (so that $\prod_{j=1}^n (x + x_j) = x^n + \sigma_1 x^{n-1} + \cdots + \sigma_n$). Consider the homomorphisms (c.f. [Lecture 6, (1.2)–(1.3)])

$$\mathrm{Fun}(\mathfrak{n}) \otimes \mathrm{Fun}(\mathfrak{g})^{\mathfrak{g}} \otimes \mathrm{Fun}(\mathfrak{n}^-) \longrightarrow \mathrm{Fun}(\mathfrak{g}), \tag{1.2}$$

$$\mathrm{Fun}(\mathfrak{n}) \otimes \mathrm{Fun}(\mathfrak{t})^{W} \otimes \mathrm{Fun}(\mathfrak{n}^-) \longrightarrow \mathrm{Fun}(\mathfrak{g}), \tag{1.3}$$

which are induced by the decomposition $\mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{t} \oplus \mathfrak{n}$.

(1) For each $1 < i \leqslant n$, find the unique element $\tilde{\sigma}_i \in \mathrm{Fun}(\mathfrak{g})^{\mathfrak{g}}$ corresponding to σ_i via the Chevalley isomorphism $\mathrm{Fun}(\mathfrak{g})^{\mathfrak{g}} \xrightarrow{\sim} \mathrm{Fun}(\mathfrak{t})^W$. In other words, find the unique extension of σ_i to an adjoint invariant polynomial function on $\mathfrak{g}$.
(2) For $\mathfrak{g} = \mathfrak{sl}_2$, prove that (1.2) and (1.3) are both injective and have the same image.
(3) For $\mathfrak{g} = \mathfrak{sl}_3$, prove that $\tilde{\sigma}_3$ is contained in the image of (1.2) but not in the image of (1.3).

Solution. (1) We need to construct $(\tilde{\sigma}_i \colon \mathfrak{g} \to k) \in \mathrm{Fun}(\mathfrak{g})^{\mathfrak{g}}$. For this, given $X \in \mathfrak{g}$, the construction is given by its characteristic polynomial

$$\det(\lambda \cdot \mathrm{id} - X) = \lambda^n + \sum_{i=1}^n (-1)^i \tilde{\sigma}_i(X) \cdot \lambda^{n-i} \in k[\lambda].$$

It is clear that coefficients $\tilde{\sigma}_i \in \mathrm{Fun}(\mathfrak{g})$, and $\tilde{\sigma}_i|_{\mathfrak{t}} = \sigma_i \colon \mathfrak{t} \to k$. Then it remains to check the $\mathfrak{g}$-adjoint invariance. But note that the $\mathfrak{g}$-invariance is the same as G-invariance; then this can be proved by noticing that the characteristic polynomial is invariant under adjoint action, i.e. $\det(\lambda \cdot \mathrm{id} - UXU^{-1}) = \det(\lambda \cdot \mathrm{id} - X)$ for $U \in \mathfrak{g}$, and so also is $\tilde{\sigma}_i(X)$.

(2) Fix an isomorphism $\mathfrak{g} \simeq \mathfrak{g}^*$ that induces $\mathfrak{t} \simeq \mathfrak{t}^*$. For $\mathfrak{g} = \mathfrak{sl}_2$ with the standard basis e, f, h, recall from [Lecture 6, Example 1.16] that

$$\mathrm{Fun}(\mathfrak{g})^{\mathfrak{g}} \simeq \mathrm{Sym}(\mathfrak{g})^{\mathfrak{g}} = k[\Omega], \quad \mathrm{Fun}(\mathfrak{t})^W \simeq \mathrm{Sym}(\mathfrak{t})^W = k[h^2].$$

Here $\Omega = h^2 + 4ef$ is the image of the Casimir element. Then we can respectively identify (1.2) and (1.3) with

$$k[e] \otimes k[\Omega] \otimes k[f] \longrightarrow k[e, f, h], \quad f_1(e) \otimes f_2(\Omega) \otimes f_3(f) \longmapsto f_1(e) f_2(\Omega) f_3(f)$$

and

$$k[e] \otimes k[h^2] \otimes k[f] \longrightarrow k[e, f, h], \quad f_1(e) \otimes f_2(h^2) \otimes f_3(f) \longmapsto f_1(e) f_2(\Omega) f_3(f).$$

Then using the property of tensor product both maps are clearly injective. Moreover, the images are respectively given by $k[e, \Omega, f]$ and $k[e, h^2, f]$, which are the same in $k[e, f, h]$.

(3) We know from part (1) that $\tilde{\sigma}_3 \in \mathrm{Fun}(\mathfrak{g})^{\mathfrak{g}}$, and is thus contained in the image of (1.2). If we write $X = (x_{ij}) \in \mathfrak{sl}_3$ as a 3×3-matrix then $\tilde{\sigma}_3(X)$ is a polynomial in all entries x_{ij}. If $\tilde{\sigma}_3$ lies in the image of (1.3) then $\tilde{\sigma}_3(X) = \det(X)$ must be a symmetric polynomial in x_{11}, x_{22}, x_{33} (with regarding other x_{ij} with $i \neq j$ as constant coefficients), which is impossible by elementary computation of $\det(\lambda I - X)$. □

Problem 3.2 (Lecture 6, Exercise 3.1)**.** Consider $\mathfrak{g} = \mathfrak{sl}_3$ and its standard Borel $\mathfrak{b}$ and Cartan subalgebras $\mathfrak{t}$. Let α_1 and α_2 be the two simple positive roots.

(1) Prove: the elements in $W \cdot 0$ are given by

0	$s_1 \cdot 0$	$s_2 \cdot 0$	$s_1 s_2 \cdot 0$	$s_2 s_1 \cdot 0$	$w_0 \cdot 0$
0	$-\alpha_1$	$-\alpha_2$	$-2\alpha_1 - \alpha_2$	$-\alpha_1 - 2\alpha_2$	$-2\alpha_1 - 2\alpha_2$

Here $w_0 = s_1 s_2 s_1 = s_2 s_1 s_2$ is the *longest element* in W.
(2) Prove: $M_{-2\alpha_1 - 2\alpha_2}$ is irreducible.
(3) Prove: $M_{-\alpha_1 - 2\alpha_2}$ contains $M_{-2\alpha_1 - 2\alpha_2}$ as a submodule[5] and the quotient is irreducible[6]. Deduce $\mathrm{length}(M_{-\alpha_1 - 2\alpha_2}) = 2$ and

$$[M_{-\alpha_1 - 2\alpha_2} : L_{-\alpha_1 - 2\alpha_2}] = [M_{-\alpha_1 - 2\alpha_2} : L_{-2\alpha_1 - 2\alpha_2}] = 1.$$

[5]Hint: [Lecture 5, Lemma 1.3].

[6]Hint: Count the dimension of the $(-2\alpha_1 - 2\alpha_2)$-weight subspace of $M_{-\alpha_1 - 2\alpha_2}$.

Solution. (1) Since the positive roots of $\mathfrak{sl}_3$ are exactly $\alpha_1, \alpha_2, \alpha_1 + \alpha_2$, we have the half-sum $\rho = \frac{1}{2}(\alpha_1 + \alpha_2 + (\alpha_1 + \alpha_2)) = \alpha_1 + \alpha_2$. Recall that the W-dotted action is given by

$$w \cdot \lambda = w(\lambda + \rho) - \rho, \quad w \in W.$$

Thus to compute $w \cdot 0$ we first notice that

$$s_1(\alpha_1) = -\alpha_1, \quad s_1(\alpha_2) = \alpha_1 + \alpha_2,$$
$$s_2(\alpha_1) = \alpha_1 + \alpha_2, \quad s_2(\alpha_2) = -\alpha_2.$$

Using these we immediately get $s_1 \cdot 0 = -\alpha_1$ and $s_2 \cdot 0 = -\alpha_2$. Also, $s_1 s_2 \cdot 0 = s_1(s_2(\rho)) - \rho = s_1(\alpha_1) - (\alpha_1 + \alpha_2) = -2\alpha_1 - \alpha_2$, and $s_2 s_1 \cdot 0 = \alpha_1 - 2\alpha_2$ by a similar computation. It then follows that $w_0 \cdot 0 = s_2(s_1 s_2(\rho)) - \rho = s_2(-2\alpha_1 - \alpha_2) - (\alpha_1 + \alpha_2) = -2\alpha_1 - 2\alpha_2$.

(2) Recall from [Lecture 6, Corollary 2.3] that if $M_{-2\alpha_1-2\alpha_2}$ contains an irreducible submodule of form L_μ, then

$$\mu = w \cdot (-2\alpha_1 - 2\alpha_2) \preceq -2\alpha_1 - 2\alpha_2.$$

Note that this implies $\mu \in W \cdot 0$. On the other hand, since part (1) has given all elements in $W \cdot 0$, we deduce $\mu = -2\alpha_1 - 2\alpha_2$ and thus $M_{-2\alpha_1-2\alpha_2} = L_{-2\alpha_1-2\alpha_2}$, which proves the irreducibility.

(3) From the proof of part (1) we see $-2\alpha_1 - 2\alpha_2 = s_1(-\alpha_1 - 2\alpha_2)$, and s_1 is defined by the simple root $\alpha_1 \in \Delta$. Also, it is clear that $\langle(-\alpha_1 - 2\alpha_2) + \rho, \check{\alpha}_1\rangle = \langle -\alpha_2, \alpha_1\rangle \in \mathbb{Z}^{\geqslant 0}$. Then $M_{-\alpha_1-2\alpha_2}$ contains $M_{-2\alpha_1-2\alpha_2}$ as a submodule by [Lecture 5, Lemma 1.3].

To show the irreducibility of $M_{-\alpha_1-2\alpha_2}/M_{-2\alpha_1-2\alpha_2}$, suppose N is a proper submodule of $M_{-\alpha_1-2\alpha_2}$ containing $M_{-2\alpha_1-2\alpha_2}$. We claim that

$$\dim M_{-\alpha_1-2\alpha_2}^{\mathrm{wt}=-2\alpha_1-2\alpha_2} = 1.$$

Indeed, if we write $f_1, f_2, f_3 \in \mathfrak{n}^-$ for the root vectors corresponding to $\alpha_1, \alpha_2, \alpha_1+\alpha_2$, respectively, then they generate a PBW basis of $U(\mathfrak{n}^-)$. Also, if v_μ is a weight vector of weight μ, then $f_i \cdot v_\mu = v_{\mu-\alpha_i}$ for $i = 1, 2$. Known this, we see as in our case there is no more weights between $-\alpha_1 - 2\alpha_2$ and $-2\alpha_1 - 2\alpha_2$ with respect to the order $\preceq$, the vectors in $M_{-\alpha_1-2\alpha_2}$ of weight $-2\alpha_1 - 2\alpha_2$ can appear exactly at once in $M_{-2\alpha_1-2\alpha_2}$ through the action of $f_1 \in U(\mathfrak{n}^-)$. As $M_{-2\alpha_1-2\alpha_2}$ is free of rank one over $U(\mathfrak{n}^-)$, we have proved claim. Granting the claim, note that the highest weight of N is at least $-2\alpha_1 - 2\alpha_2$, which by our assumption on N implies $N = M_{-2\alpha_1-2\alpha_2}$. Therefore, we conclude that

$$M_{-\alpha_1-2\alpha_2}/M_{-2\alpha_1-2\alpha_2} \simeq L_{-\alpha_1-2\alpha_2}$$

is irreducible.

Combining this with part (2), we see $M_{-\alpha_1-2\alpha_2}$ contains an irreducible submodule whose quotient is also irreducible, and hence $\mathrm{length}(M_{-\alpha_1-2\alpha_2}) = 2$; here both $[M_{-\alpha_1-2\alpha_2} : L_{-\alpha_1-2\alpha_2}]$ and $[M_{-\alpha_1-2\alpha_2} : L_{-2\alpha_1-2\alpha_2}]$ are forced to be 1. $\square$

Problem 3.3 (Lecture 6, Exercise 3.3)**.** We continue with the case $\mathfrak{g} = \mathfrak{sl}_3$.

(1) Prove: M_0 contains $M_{-\alpha_1}$ and $M_{-\alpha_2}$ as submodules and $[M_0 : L_{-\alpha_1}] = [M_0 : L_{-\alpha_2}] = 1$.
(2) Prove: $M_{-\alpha_2}$ contains $M_{-2\alpha_1-\alpha_2}$ as a submodule and $[M_{-\alpha_2} : L_{-2\alpha_1-\alpha_2}] = 1$.
(3) Prove: there exists a (unique) dotted arrow making the following diagram commutes[7]

$$\begin{tikzcd} & M_{-\alpha_1} \arrow[r, "\subset"] & M_0 & \\ M_{-2\alpha_1-\alpha_2} \arrow[ur, dashed] \arrow[rrr, "\subset"] & & & M_{-\alpha_2}. \arrow[ul, "\subset"'] \end{tikzcd}$$

Solution. (1) The argument is the same as that in Problem 3.2(3). For the first assertion, the result follows from $-\alpha_i = s_i \cdot 0$ and $\langle 0 + \rho, \check{\alpha}_i\rangle = 1 \in \mathbb{Z}^{\geqslant 0}$ for $i = 1, 2$. Further, since there is no more weights between 0 and $-\alpha_i$, the quotient $M_0/M_{-\alpha_i} \simeq L_0$ is irreducible. Thus, for the second assertion, as $L_{-\alpha_i}$ and L_0 are distinct, we get $[M_0 : L_{-\alpha_i}] = [M_{-\alpha_i} : L_{-\alpha_i}] = 1$ for $i = 1, 2$.

[7]Hint: Show $f_1^2 f_2 \cdot v_0 = u \cdot v_{-\alpha_1}$ for some u. Here v_0 is the highest weight of M_0, $v_{-\alpha_1} = f_1 \cdot v_0$ is the highest weight of $M_{-\alpha_1}$, and $f_i \in \mathfrak{n}^-$ is the root vector corresponding to α_i.

(2) As before, since $s_1 \cdot (-\alpha_2) = -2\alpha_1 - \alpha_2$ and $\langle -\alpha_2 + \rho, \check{\alpha}_1 \rangle = \langle \alpha_1, \alpha_1 \rangle = 2 \in \mathbb{Z}^{\geqslant 0}$, we know that $M_{-\alpha_2}$ contains $M_{-2\alpha_1-\alpha_2}$ as a submodule. Similar to Problem 3.2(3), we have that

$$\dim M_{-\alpha_1}^{\mathrm{wt}=-2\alpha_1-\alpha_2} = 1.$$

This proves $[M_{-\alpha_2} : L_{-2\alpha_1-\alpha_2}] = 1$.

(3) For each $\lambda \in W \cdot 0$, let v_λ be the highest weight vector of M_λ. Following the hint, it suffices to show $f_1^2 f_2 \cdot v_0 = u \cdot v_{-\alpha_1} = u f_1 \cdot v_0$ for some $u \in U(\mathfrak{n}^-)$. For this, notice that

$$f_1^2 f_2 = f_1([f_1, f_2] + f_2 f_1) = f_1[f_1, f_2] + f_1 f_2 f_1.$$

Also notice that both $f_1[f_1, f_2] \cdot v_0$ and $[f_1, f_2] f_1 \cdot v_0$ are highest weight vectors of $M_{-2\alpha_1-\alpha_2}$, because $[f_1, f_2]$ sends a vector of weight v_μ to that of weight $\mu - (\alpha_1 + \alpha_2)$. So there exists $\lambda \in k^\times$ such that $f_1[f_1, f_2] \cdot v_0 = \lambda [f_1, f_2] f_1 \cdot v_0$. It then leads to

$$f_1^2 f_2 \cdot v_0 = (\lambda [f_1, f_2] f_1 + f_1 f_2 f_1) \cdot v_0,$$

and taking $u = \lambda [f_1, f_2] + f_1 f_2$ completes the proof.[8] □

Problem 3.4 (Lecture 6, Exercise 3.4)**.** We continue with the case $\mathfrak{g} = \mathfrak{sl}_3$. Prove: for $\lambda, \mu \in W \cdot 0$, $[M_\lambda : L_\mu] \neq 0$ if and only if $\lambda \succeq \mu$.

Solution. It is clear that $[M_\lambda : L_\mu] \neq 0$ implies $\lambda \succeq \mu$. Conversely, we need to apply an enumeration and use the known results in Problems 3.2 and 3.3. We need to show that each map in the following diagram is an inclusion of modules, because these maps are exactly all maps of the form $M_\mu \to M_\lambda$ with $\mu \preceq \lambda$.

$$\begin{tikzcd}
 & M_{-2\alpha_1-\alpha_2} \arrow[r] \arrow[ddr] & M_{-\alpha_1} \arrow[dr] & \\
M_{-2\alpha_1-2\alpha_2} \arrow[ur] \arrow[dr] & & & M_0 \\
 & M_{-\alpha_1-2\alpha_2} \arrow[r] \arrow[uur] & M_{-\alpha_2} \arrow[ur] &
\end{tikzcd}$$

For this, it is known that

- $M_{-\alpha_i} \hookrightarrow M_0$ for $i = 1, 2$ by Problem 3.3(1);
- $M_{-2\alpha_1-\alpha_2} \hookrightarrow M_{-\alpha_2}$ by Problem 3.3(2);
- $M_{-2\alpha_1-\alpha_2} \hookrightarrow M_{-\alpha_1}$ by Problem 3.3(3);
- $M_{-2\alpha_1-2\alpha_2} \hookrightarrow M_{-\alpha_1-2\alpha_2}$ by Problem 3.2(3).

So it remains to show $M_{-\alpha_1-2\alpha_2} \hookrightarrow M_{-\alpha_1}$, $M_{-\alpha_1-2\alpha_2} \hookrightarrow M_{-\alpha_2}$, and $M_{-2\alpha_1-2\alpha_2} \hookrightarrow M_{-2\alpha_1-\alpha_2}$. But these follow from similar arguments as in 3.3(2), 3.3(3), and 3.2(3), respectively. □

Problem 3.5 (Lecture 7, Exercise 4.20)**.** For any weight $\lambda \in \mathfrak{t}^*$, let $\mathcal{O}_\lambda \subset \mathcal{O}$ be the full subcategory containing those objects M whose composition factors are of the form L_μ for $\mu \in W_{[\lambda]} \cdot \lambda$. Prove:

(1) Each $\mathcal{O}_\lambda$ is indecomposable.

(2) For $M \in \mathcal{O}$, suppose we have a decomposition $M \simeq M_1 \oplus M_2$ as $\mathfrak{t}$-modules such that the set $\mathrm{wt}(M_1) - \mathrm{wt}(M_2) := \{\lambda_1 - \lambda_2 \mid \lambda_i \in \mathrm{wt}(M_i)\}$ has empty intersection with Λ_r, then this is also a decomposition of $\mathfrak{g}$-modules.

(3) For any central character χ of $Z(\mathfrak{g})$, we have a direct sum decomposition

$$\mathcal{O}_\chi \simeq \bigoplus_{\substack{\lambda \in \varpi^{-1}(\chi) \\ \text{dot-antidominant}}} \mathcal{O}_\lambda.$$

[8] In fact, one can even deduce the formula of u in a more explicit sense. As in Problem 3.2(3), let f_1, f_2, f_3 be (standard) generators of PBW basis of $U(\mathfrak{n}^-)$ corresponding to simple roots $\alpha_1, \alpha_2, \alpha_1 + \alpha_2$, respectively. Then there are relations turning out to be $[f_1, f_2] = f_3$ and $[f_1, f_3] = 0$. Using these together with some elementary computation, we get $u = f_1 f_2 + f_3$, and hence $\lambda = 1$ in the original proof.

(4) Conclude that
$$\mathcal{O} \simeq \bigoplus_{\lambda \text{ dot-antidominant}} \mathcal{O}_\lambda.$$

Solution. (1) The proof uses the same idea as [Lecture 7, Proposition 4.19]. We may assume λ is dot-antidominant and suppose $\mathcal{O}_\lambda \simeq \mathcal{O}_1 \oplus \mathcal{O}_2$. Recall any Verma module M_μ is indecomposable because it has a unique irreducible quotient L_μ. Moreover, if $\mu \in W_{[\lambda]} \cdot \lambda$, then $M_\lambda \subset M_\mu$ as a submodule by [Lecture 7, Corollary 4.15(2)]. Note that M_λ lies in either $\mathcal{O}_1$ or $\mathcal{O}_2$, so all M_μ are simultaneously contained in one of $\mathcal{O}_1$ and $\mathcal{O}_2$, and thus either $\mathcal{O}_1 \simeq 0$ or $\mathcal{O}_2 \simeq 0$ holds.

(2) It suffices to show that M_1 is a $\mathfrak{g}$-module, and the same argument applies to M_2. For this, taking $v \in M_1$, we need that $x \cdot v \in M_1$ for all root vectors $x \in \mathfrak{g}$. If this is not true then $x_0 \cdot v \in M_2$ for some root vector $x_0 \in \mathfrak{g}$ and then
$$(\text{weight of } v) - (\text{weight of } x_0 \cdot v) \in (\mathrm{wt}(M_1) - \mathrm{wt}(M_2)) \cap \Lambda_r,$$
which is impossible by assumption that $(\mathrm{wt}(M_1) - \mathrm{wt}(M_2)) \cap \Lambda_r = \emptyset$. Thus both M_1 and M_2 can be regarded as $\mathfrak{g}$-modules.

(3) Considering W-dotted action on a fixed element $\mu \in \varpi^{-1}(\chi) \subset \mathfrak{t}^*$, we have a decomposition
$$W \cdot \mu = \bigsqcup_{\lambda \text{ dot-antidominant}} W_{[\lambda]} \cdot \lambda.$$
Recall that $\mathcal{O}_\chi$ is the full subcategory of $\mathcal{O}$ supported on χ. Then for each object $M \in \mathcal{O}_\chi$, the decomposition of $W \cdot \mu$ above leads to a decomposition of $\mathfrak{t}$-modules
$$M = \bigoplus_{\substack{\lambda \in \varpi^{-1}(\chi) \\ \text{dot-antidominant}}} M_{[\lambda]}.$$
Notice that for two distinct dot-antidominant weights $\lambda, \lambda' \in \varpi^{-1}(\chi)$, we obtain $(\mathrm{wt}(M_{[\lambda]}) - \mathrm{wt}(M_{[\lambda']})) \cap \Lambda_r = \emptyset$ by decomposition of $W \cdot \mu$ together with the definition of $W_{[\lambda]}$; in particular, $\mathrm{wt}(M_{[\lambda]}) \cap \mathrm{wt}(M_{[\lambda']}) = \emptyset$ as $0 \in \Lambda_r$. Then by part (2) the decomposition of M above upgrades to a decomposition of $\mathfrak{g}$-modules, with the property that $\mathrm{Hom}_{\mathcal{O}_\chi}(M_{[\lambda]}, M_{[\lambda']}) = 0$. So the desired decomposition of categories follows.

(4) From the block decomposition we have that
$$\mathcal{O} \simeq \bigoplus_{\chi \in \operatorname{Spec} Z(\mathfrak{g})} \mathcal{O}_\chi.$$
Note that for each $\chi \in \operatorname{Spec} Z(\mathfrak{g})$ the set of dot-antidominant weights in $\varpi^{-1}(\chi)$ is nonempty, because $\varpi \colon \mathfrak{t}^* \to \operatorname{Spec} Z(\mathfrak{g}) \simeq \mathfrak{t}^* /\!\!/ W$ is surjective. So it makes sense to apply part (3) to deduce
$$\mathcal{O} \simeq \bigoplus_{\chi \in \operatorname{Spec} Z(\mathfrak{g})} \bigoplus_{\substack{\lambda \in \varpi^{-1}(\chi) \\ \text{dot-antidominant}}} \mathcal{O}_\lambda \simeq \bigoplus_{\lambda \text{ dot-antidominant}} \mathcal{O}_\lambda.$$

$\square$

Homework 4

Problem 4.1 (Lecture 8, Exercise 1.6)**.** Prove the following.

(1) For $w, w' \in W$ such that $\ell(w) + \ell(w') = \ell(ww')$, we have
$$\delta_w \delta_{w'} = \delta_{ww'}.$$

(2) For $w \in W$ and $s \in S$, we have
$$\delta_w \delta_s = \begin{cases} \delta_{ws} & \text{if } w < ws, \\ (v^{-1} - v)\delta_w + \delta_{ws} & \text{if } w > ws. \end{cases}$$
and
$$\delta_s \delta_w = \begin{cases} \delta_{sw} & \text{if } w < sw, \\ (v^{-1} - v)\delta_w + \delta_{sw} & \text{if } w > ws. \end{cases}$$

Solution. (1) Recall that if $w = s_{i_1} \cdots s_{i_k}$ with $k = \ell(w)$ is a reduced decomposition of w, then $\delta_w = \delta_{s_{i_1}} \cdots \delta_{s_{i_k}}$. Provided that $\ell(w) + \ell(w') = \ell(ww')$, we see a reduced decomposition of ww' can be constructed by concatenating those of w and w'. Then
$$\delta_w \delta_{w'} = \delta_{ww'}.$$

(2) We first compute $\delta_w \delta_s$. Suppose $w < ws$ then $\ell(ws) = \ell(w) + 1 = \ell(w) + \ell(s)$. Then by part (1) the result $\delta_w \delta_s = \delta_{ws}$ follows. Whenever $w > ws$ we may write $w = w's$ for some $w' \in W$; note that if this is the case then $ws = w's^2 = w'$, and hence $\delta_w = \delta_{w'} \delta_s$ with $\delta_{w'} = \delta_{ws}$. Recall the inverse formula
$$\delta_s^{-1} = \delta_s + (v - v^{-1})$$
which is equivalent to $\delta_s^2 = (v^{-1} - v)\delta_s + 1$. Combining the ingredients above we obtain that
$$\delta_w \delta_s = \delta_{w'} \delta_s^2 = \delta_{w'}((v^{-1} - v)\delta_s + 1) = (v^{-1} - v)\delta_{w'}\delta_s + \delta_{w'} = (v^{-1} - v)\delta_w + \delta_{ws}.$$
This proves the formula of $\delta_w \delta_s$, and the case of $\delta_s \delta_w$ follows from a similar argument. □

Problem 4.2 (Lecture 8, Exercise 4.8)**.** For any $\lambda \in \mathfrak{t}^*$, prove

(1) The surjection $P_\lambda \twoheadrightarrow L_\lambda$ factors as $P_\lambda \to M_\lambda \to L_\lambda$.
(2) The obtained map $P_\lambda \to M_\lambda$ is surjective and exhibits P_λ as a projective cover of M_λ.

Solution. (1) By definition P_λ is a projective cover of L_λ; in particular, it is a projective module. On the other hand, $M_\lambda \twoheadrightarrow L_\lambda$ is the canonical projection in Verma module, so $P_\lambda \twoheadrightarrow L_\lambda$ factors through M_λ by the universal property of projective module P_λ.

(2) Recall that M_λ is generated by a highest weight vector v_λ; so to show that $P_\lambda \to M_\lambda$ is surjective, it boils down to figuring out the preimage of v_λ in P_λ. Since $P_\lambda \twoheadrightarrow L_\lambda$ is surjective, if we write $\overline{v}_\lambda$ for the image of v_λ along $M_\lambda \twoheadrightarrow L_\lambda$, then there exists some $w_\lambda \in P_\lambda$ mapping to $\overline{v}_\lambda$. Note that w_λ is nonzero of weight λ, and hence mapped to a highest weight vector $v'_\lambda \in M_\lambda$; on the other hand, the subspace in M_λ spanned by highest weight vector is 1-dimensional, so $v'_\lambda = c \cdot v_\lambda$ for some $c \in k^\times$, which deduces that v_λ is the image of $c^{-1} w_\lambda$. Therefore, $P_\lambda \twoheadrightarrow M_\lambda$ is surjective.

To exhibit P_λ as a projective cover of M_λ, notice that the surjectivity result above can be reinterpreted as follows: P_λ surjects to M_λ if and only if it surjects to L_λ. We need to show that any proper submodule of P_λ cannot be mapped to M_λ by any surjection. For this, if $N \subset P_\lambda$ is a proper submodule with $N \twoheadrightarrow M_\lambda$, then we also have $N \twoheadrightarrow L_\lambda$, which contradicts to the fact P_λ is a projective cover of L_λ. □

Problem 4.3 (Lecture 8, Exercise 4.12)**.** For $\lambda \in \mathfrak{t}^*$, let $P \twoheadrightarrow L_\lambda$ be the surjection constructed in the proof of [Lecture 8, Theorem 4.3], i.e., P represents the functor
$$\mathcal{O}_\chi \longrightarrow \mathsf{Vect}, \quad M \longmapsto M^{\mathrm{wt}=\lambda}.$$
Prove:

(1) This map factors as $P \to P_\lambda \twoheadrightarrow L_\lambda$. Moreover, $P \to P_\lambda$ is surjective.

(2) For $\mathfrak{g} = \mathfrak{sl}_2$, the obtained map $P \to P_\lambda$ happens to be an isomorphism[9].
(3) In general, $P \to P_\lambda$ is not an isomorphism[10].

Solution. (1) Recall from proof of [Lecture 8, Theorem 4.3] that P is projective, so the factorization follows from the universal property, provided that $P_\lambda \twoheadrightarrow L_\lambda$ is a projective cover by Problem 4.2. Notice that the surjection $P \twoheadrightarrow L_\lambda$ induces another surjection $\operatorname{im}(P \to P_\lambda) \twoheadrightarrow L_\lambda$, and then $P_\lambda = \operatorname{im}(P \to P_\lambda)$ because P_λ is essential as a projective cover. This shows $P = \operatorname{im}(P \to P_\lambda)$, namely $P \twoheadrightarrow P_\lambda$ is surjective.

(2) As P is projective, by part (1) we get $P \cong P_\lambda \oplus K$ with $K \coloneqq \ker(P \twoheadrightarrow P_\lambda)$. Note that the injection $P_\lambda \hookrightarrow P$ induces $\operatorname{Hom}_{\mathcal{O}_\chi}(P_\lambda, M) \hookrightarrow \operatorname{Hom}_{\mathcal{O}_\chi}(P, M)$ for all $M \in \mathcal{O}_\chi$. It then suffices to show

$$(*) \qquad \dim \operatorname{Hom}_{\mathcal{O}_\chi}(P, M) = \dim \operatorname{Hom}_{\mathcal{O}_\chi}(P_\lambda, M),$$

because if this is true then the two spaces are the same and Yoneda lemma deduces $P = P_\lambda$. To prove $(*)$, since P represents the functor $M \mapsto M^{\mathrm{wt}=\lambda}$, we have

$$\operatorname{Hom}_{\mathcal{O}_\chi}(P, M) = M^{\mathrm{wt}=\lambda}.$$

On the other hand, by [Lecture 8, Corollary 4.9], we have

$$\dim \operatorname{Hom}_{\mathcal{O}_\chi}(P_\lambda, M) = [M : L_\lambda].$$

So it remains to prove $\dim M^{\mathrm{wt}=\lambda} = [M : L_\lambda]$ for any simple objects $M \in \mathcal{O}_\chi$. But in the case of $\mathfrak{g} = \mathfrak{sl}_2$, this follows from the classification in [Lecture 8, Example 3.17], together with the fact that when $l, l' \in \mathbb{Z}$, the l'-weight space of M_l is 1-dimensional (so that for distinct weights $\mu \neq \lambda$ there is no vector in L_μ of weight λ).

(3) We give a counter-example for $\mathfrak{g} = \mathfrak{sl}_3$ to show the equality $\dim M^{\mathrm{wt}=\lambda} = [M : L_\lambda]$ in the proof of part (2) fails. In the context of Problem 3.2(3), with $\chi = \varpi(0)$, we have for $\lambda = -2\alpha_1 - \alpha_2$ and $M = M_0$ that $[M_0 : L_{-2\alpha_1-\alpha_2}] = 1$. This comes from combining part (1) and (2) of Problem 3.3 (which dictates that $[M_0 : L_{-\alpha_2}] = [M_{-\alpha_2} : M_{-2\alpha_1-\alpha_2}] = 1$). However, since $2\alpha_1 + \alpha_2$ admits two formulations into sum of positive roots, which are $\alpha_1 + \alpha_1 + \alpha_2$ and $\alpha_1 + (\alpha_1 + \alpha_2)$ respectively, we have $\dim M_0^{\mathrm{wt}=-2\alpha_1-\alpha_2} \geqslant 2$. Thus, $[M : L_\lambda] \neq \dim M^{\mathrm{wt}=\lambda}$. □

Problem 4.4 (Lecture 9, Exercise 1.11)**.** Let λ be a weight. Prove:

(1) If $M \in \mathcal{O}$ such that $\operatorname{wt}(M) \cap \{\mu \mid \mu \succeq \lambda\} = \emptyset$, then $\operatorname{Ext}^i_{\mathcal{O}}(M, M_\lambda^\vee) = 0$ for $i \geqslant 0$[11].
(2) $\operatorname{Ext}^i_{\mathcal{O}}(L_\lambda, M_\lambda^\vee) = 0$ for $i > 0$.
(3) Combining (1) and (2), deduce $\operatorname{Ext}^i_{\mathcal{O}}(M_\lambda, L_\mu) = 0$ and $\operatorname{Ext}^i_{\mathcal{O}}(L_\mu, M_\lambda^\vee) = 0$ for $i > 0$ and $\mu \not\prec \lambda$.

Solution. (1) For any $M \in \mathcal{O}$ satisfying $\operatorname{wt}(M) \cap \{\mu \mid \mu \succeq \lambda\} = \emptyset$, if L_μ is an irreducible subquotient of M then $\mu \not\succeq \lambda$ holds. Thus it reduces to considering $M = L_\mu$ with $\mu \not\succeq \lambda$. If $\varpi(\mu) \neq \varpi(\lambda)$, i.e. M and $M_\lambda^\vee$ come from different blocks in $\mathcal{O}$, we have $\operatorname{Ext}^i_{\mathcal{O}}(M, M_\lambda^\vee) = 0$ for $i \geqslant 0$ by [Lecture 9, Lemma 1.10]; so we may assume $\varpi(\mu) = \varpi(\lambda)$. We then consider the short exact sequence

$$0 \longrightarrow K \longrightarrow M_\mu \longrightarrow L_\mu \longrightarrow 0$$

induced by the canonical projection $M_\mu \twoheadrightarrow L_\mu$ with kernel K. Recall [Lecture 9, Corollary 1.3] that we have $\operatorname{Ext}^i_{\mathcal{O}}(M_\mu, M_\lambda^\vee) = 0$ for $i > 0$. Moreover, by [Lecture 8, Lemma 3.14] and the assumption $\mu \not\succeq \lambda$, we get $\operatorname{Hom}_{\mathcal{O}}(M_\mu, M_\lambda^\vee) = \operatorname{Ext}^0_{\mathcal{O}}(M_\mu, M_\lambda^\vee) = 0$, and consequently

$$\operatorname{Ext}^0_{\mathcal{O}}(L_\mu, M_\lambda^\vee) = 0.$$

Now it remains to show $\operatorname{Ext}^i_{\mathcal{O}}(L_\mu, M_\lambda^\vee) = 0$ for $i > 0$. For this, applying the contravariant functor $\operatorname{Hom}_{\mathcal{O}}(-, M_\lambda^\vee)$ to the short exact sequence above, we have a long exact sequence:

[9] Hint: Use [Lecture 8, Corollary 4.9].
[10] Hint: What we have learned so far can (at least) prove this for $\mathfrak{sl}_3$ and $\mathfrak{sl}_4$.
[11] Hint: **Step I.** Reduce to the case $M = L_\mu$ with $\varpi(\mu) = \varpi(\lambda)$ and $\mu \not\succeq \lambda$.
Step II. Consider $0 \to K \to M_\mu \to L_\mu \to 0$ and note that $\operatorname{wt}(K) \prec \mu$.

$$\begin{tikzcd}
 & & 0 \arrow[d, equal] & \\
\cdots \arrow[r] & \mathrm{Ext}^i_{\mathcal{O}}(L_\mu, M_\lambda^\vee) \arrow[r] & \mathrm{Ext}^i_{\mathcal{O}}(M_\mu, M_\lambda^\vee) \arrow[r] & \boxed{\mathrm{Ext}^i_{\mathcal{O}}(K, M_\lambda^\vee)} \arrow[dlll, rounded corners] \\
\boxed{\mathrm{Ext}^{i+1}_{\mathcal{O}}(L_\mu, M_\lambda^\vee)} \arrow[r] & \mathrm{Ext}^{i+1}_{\mathcal{O}}(M_\mu, M_\lambda^\vee) \arrow[r] \arrow[d, equal] & \mathrm{Ext}^{i+1}_{\mathcal{O}}(K, M_\lambda^\vee) \arrow[r] & \cdots \\
 & 0 & &
\end{tikzcd}$$

It then follows that for $i \geqslant 0$,

$$\mathrm{Ext}^i_{\mathcal{O}}(K, M_\lambda^\vee) \cong \mathrm{Ext}^{i+1}_{\mathcal{O}}(L_\mu, M_\lambda^\vee).$$

So we only need to prove $\mathrm{Ext}^i_{\mathcal{O}}(K, M_\lambda^\vee) = 0$ for $i \geqslant 0$. Since there are only finitely many irreducible objects in $\mathcal{O}_{\varpi(\mu)}$, this can be done by induction on μ with respect to $\prec$:

- ◇ If μ is dot-antidominant, then M_μ is irreducible and $K = 0$. In this case, the vanishing result obviously follows.
- ◇ Suppose $\mathrm{Ext}^i_{\mathcal{O}}(L_{\mu'}, M_\lambda^\vee) = 0$ holds for all $i \geqslant 0$ and $\mu' \prec \mu$. Notice that $K \in \mathcal{O}$ is always an extension of $L_{\mu'}$'s with $\mu' \prec \mu$. So we get $\mathrm{Ext}^i_{\mathcal{O}}(K, M_\lambda^\vee) = 0$ for $i \geqslant 0$ as desired.

This completes the induction and we conclude that $\mathrm{Ext}^i_{\mathcal{O}}(M, M_\lambda^\vee) = 0$ for $i \geqslant 0$.

(2) As in part (1), consider the short exact sequence

$$0 \longrightarrow K \longrightarrow M_\lambda \longrightarrow L_\lambda \longrightarrow 0.$$

Notice again that for each $\mu \in \mathrm{wt}(K)$ we have $\mu \prec \lambda$, so $\mathrm{wt}(K) \cap \{\mu \mid \mu \succeq \lambda\} = \emptyset$, and then by part (1) we get $\mathrm{Ext}^i_{\mathcal{O}}(K, M_\lambda^\vee) = 0$ for $i \geqslant 0$. On the other hand, as in part (1), we know $\mathrm{Ext}^i_{\mathcal{O}}(M_\lambda, M_\lambda^\vee) = 0$ for $i > 0$. Therefore, applying the contravariant functor $\mathrm{Hom}_{\mathcal{O}}(-, M_\lambda^\vee)$ to the short exact sequence above, we have a long exact sequence as follows:

$$\begin{tikzcd}
 & & & 0 \arrow[d, equal] \\
0 \arrow[r] & \mathrm{Hom}_{\mathcal{O}}(L_\lambda, M_\lambda^\vee) \arrow[r] & \mathrm{Hom}_{\mathcal{O}}(M_\lambda, M_\lambda^\vee) \arrow[r] & \mathrm{Hom}_{\mathcal{O}}(K, M_\lambda^\vee) \arrow[dlll, rounded corners] \\
\boxed{\mathrm{Ext}^1_{\mathcal{O}}(L_\lambda, M_\lambda^\vee)} \arrow[r] & \mathrm{Ext}^1_{\mathcal{O}}(M_\lambda, M_\lambda^\vee) \arrow[r] \arrow[d, equal] & \mathrm{Ext}^1_{\mathcal{O}}(K, M_\lambda^\vee) \arrow[r] \arrow[d, equal] & \cdots \\
 & 0 & 0 &
\end{tikzcd}$$

Thus, $\mathrm{Ext}^i_{\mathcal{O}}(L_\lambda, M_\lambda^\vee) = 0$ for $i > 0$ is implied by $\mathrm{Ext}^i_{\mathcal{O}}(M_\lambda, M_\lambda^\vee) = \mathrm{Ext}^{i-1}_{\mathcal{O}}(K, M_\lambda^\vee) = 0$; but note that this may not vanish at $i = 0$.

(3) Using the isomorphism $L_\mu \cong L_\mu^\vee$, the vanishing result of $\mathrm{Ext}^i_{\mathcal{O}}(M_\lambda, L_\mu)$ follows from that of $\mathrm{Ext}^i_{\mathcal{O}}$ by duality. Thus it suffices to check $\mathrm{Ext}^i_{\mathcal{O}}(L_\mu, M_\lambda^\vee) = 0$. If $\mu \prec \lambda$ then $M = L_\lambda$ satisfies the condition of (1), otherwise $\mu = \lambda$ and we are in the case of (2), and in both cases we have the desired vanishing result. □

Problem 4.5 (Lecture 9, Exercise 3.18)**.** Let μ be any dot-antidominant integral weight. Prove[12]:

(1) For any $w \in W$, $(\Xi_\mu : M_{w\cdot\mu}) = 1$ and there is a surjection $\Xi_\mu \twoheadrightarrow M_\mu$.
(2) For any $w \in W$, $(P_\mu : M_{w\cdot\mu}) \geqslant 1$ and there is a surjection $P_\mu \twoheadrightarrow M_\mu$.
(3) There exists an isomorphism $\Xi_\mu \simeq P_\mu$ compatible with the surjections to M_μ.
(4) For any $w \in W$, $[M_{w\cdot\mu} : L_\mu] = 1$[13].

[12]Hint: [Lecture 9, Lemma 3.5] for (1); [Lecture 9, Theorem 2.2] and [Corollary 4.15, Lecture 7] for (2). For (3), first find a surjection $\Xi_\mu \twoheadrightarrow P_\mu$ and then use (1) and (2).

[13]See [Gai05, Proposition 4.20] for a different proof of this fact.

Solution. (1) By [Lecture 9, Lemma 3.5], if ν is the unique dominant integral weight in $W(\mu-(-\rho)) = W(\mu+\rho)$, then $\Xi_\mu \coloneqq T^{\mu}_{-\rho}(L_{-\rho}) \cong T^{\mu}_{-\rho}(M_{-\rho}) \in \mathcal{O}_{\varpi(\mu)}$ admits a standard filtration and we have the multiplicity

$$(\Xi_\mu : M_{w\cdot\mu}) = (T^{\mu}_{-\rho}(M_{-\rho}) : M_{w\cdot\mu}) = \dim L_\nu^{\mathrm{wt}=w\cdot\mu-(-\rho)} = 1.$$

Here the last equality follows from $w\cdot\mu+\rho \in W(\nu)$ by our construction. In particular, taking $w=1$, we deduce

$$\dim \mathrm{Hom}_{\mathcal{O}}(\Xi_\mu, M_\mu^\vee) = (\Xi_\mu : M_\mu) = 1.$$

Consequently, there exists a non-zero map $\phi\colon \Xi_\mu \to M_\mu^\vee$ in $\mathcal{O}$. As μ is dot-antidominant, we see $M_\mu \cong M_\mu^\vee$ is irreducible, and hence Ξ_μ must surjects onto M_μ through $\Xi_\mu \to M_\mu^\vee \simeq M_\mu$, which is the desired surjection.

(2) Using BGG reciprocity [Lecture 9, Theorem 2.2], we have

$$(P_\mu : M_{w\cdot\mu}) = [M_{w\cdot\mu} : L_\mu].$$

To compute the right hand side, by [Lecture 7, Corollary 4.15] and that μ is dot-antidominant integral, $L_\mu \cong M_\mu \subset M_{w\cdot\mu}$ as a submodule. So we get $[M_{w\cdot\mu} : L_\mu] \geqslant 1$ and hence $(P_\mu : M_{w\cdot\mu}) \geqslant 1$. Also, the surjection $P_\mu \twoheadrightarrow M_\mu$ is constructed as in Problem 4.2(2).

(3) Since we have the surjection $\Xi_\mu \twoheadrightarrow M_\mu$ from part (1) and Ξ_μ is by definition a projective object, the surjection $P_\mu \twoheadrightarrow M_\mu$ induces a factorization diagram

$$\begin{tikzcd} \Xi_\mu \arrow[r, dashed, "\eta"] \arrow[dr, two heads] & P_\mu \arrow[d, two heads] \\ & M_\mu \end{tikzcd}$$

and then we get a map $\eta\colon \Xi_\mu \to P_\mu$ compatible with the surjections to M_μ.

Now it remains to show that η is an isomorphism. Recall from Problem 4.2 that P_μ is a projective cover of M_μ, so η must be surjective by the same argument thereof. To show the injectivity, define the weight submodule $K \coloneqq \ker(\Xi_\mu \twoheadrightarrow P_\mu) \subset \Xi_\mu$. Then by (1) and (2), for any $w \in W$,

$$(\Xi_\mu : M_{w\cdot\mu}) = 1 \leqslant (P_\mu : M_{w\cdot\mu}) \implies (K : M_{w\cdot\mu}) = 0.$$

This makes sense because K admits a standard filtration by [Lecture 9, Lemma 1.8]. Therefore, we get $K=0$, and $\eta\colon \Xi_\mu \xrightarrow{\sim} P_\mu$ is an isomorphism.

(4) It follows from part (3) directly that

$$[M_{w\cdot\mu} : L_\mu] = (P_\mu : M_{w\cdot\mu}) = (\Xi_\mu : M_{w\cdot\mu}) = 1.$$

$\square$

Homework 5

Problem 5.1 (Lecture 10, Exercise 3.3)**.** Let X be an affine smooth k-scheme. Prove: Any k-derivation $\mathcal{O}(X) \to \mathcal{O}(X)$ is a differential operator of order 1, and the obtained map

$$\mathcal{O}(X) \oplus \mathcal{T}(X) \longrightarrow \mathsf{F}^{\leqslant 1}\mathcal{D}(X)$$

is an isomorphism.

Solution. Let $\partial\colon \mathcal{O}(X) \to \mathcal{O}(X)$ be any k-derivation, regarded as an element of $\mathcal{T}(X)$. Then for any $f, g \in \mathcal{O}(X)$, we have

$$[\partial, f](g) = \partial(fg) - f \cdot \partial(g) = \partial(f) \cdot g.$$

Here the first equality above is by definition of Lie bracket, and the second is because ∂ satisfies Leibniz rule. Since multiplying by $\partial(f) \in \mathcal{O}(X)$ is known to be a differential operator of order 0, we see ∂ is a differential operator of order 1.

To prove the provided map is an isomorphism, we need to show that each element of $\mathsf{F}^{\leqslant 1}\mathcal{D}(X)$ can be uniquely written as a sum $f + \partial$ with $f \in \mathcal{O}(X)$ and $\partial \in \mathcal{T}(X)$. From the argument above we know $f \in \mathsf{F}^{\leqslant 0}\mathcal{D}(X)$ and $\partial \in \mathsf{F}^{\leqslant 1}\mathcal{D}(X)$, so that $f + \partial \in \mathsf{F}^{\leqslant 1}\mathcal{D}(X)$. Thus the following k-linear map makes sense:

$$\Phi\colon \mathcal{O}(X) \oplus \mathcal{T}(X) \longrightarrow \mathsf{F}^{\leqslant 1}\mathcal{D}(X), \quad (f, \partial) \longmapsto f + \partial.$$

Now it suffices to check the bijectivity of Φ.

For the injectivity, let $f + \partial = 0$ in $\operatorname{im}\Phi$, then $f \cdot g + \partial(g) = 0$ for all $g \in \mathcal{O}(X)$. In particular, when $g = \mathbb{1}$ we obtain

$$f + \partial(\mathbb{1}) = 0.$$

Here $\partial \in \mathcal{T}(X)$ satisfies Leibniz rule, so $\partial(\mathbb{1}) = \partial(\mathbb{1} \cdot \mathbb{1}) = \mathbb{1} \cdot \partial(\mathbb{1}) + \mathbb{1} \cdot \partial(\mathbb{1}) = 2\partial(\mathbb{1})$, which implies $\partial(\mathbb{1}) = 0$. It follows that $f = 0$, and thus $\partial = 0$ as well, which proves the injectivity.

For the surjectivity, let $D \in \mathsf{F}^{\leqslant 1}\mathcal{D}(X)$. Caution that D does not need to satisfy Leibniz rule. For any $f \in \mathcal{O}(X)$, the operator

$$[D, f]\colon \mathcal{O}(X) \longrightarrow \mathcal{O}(X), \quad s \longmapsto D(fs) - f \cdot D(s)$$

is of order 0, so we get $[D, f](\mathbb{1}) \in \mathcal{O}(X)$, and $[D, f](s) = [D, f](\mathbb{1}) \cdot s$ holds for all $s \in \mathcal{O}(X)$. In particular, we take $s = \mathbb{1} \in \mathcal{O}(X)$ to deduce

$$[D, f](\mathbb{1}) = D(f) - f \cdot D(\mathbb{1}) \in \mathcal{O}(X).$$

Using this formula of $[D, f]$, for any $g \in \mathcal{O}(X)$, we deduce

$$D(fg) - f \cdot D(g) = [D, f](g) = [D, f](\mathbb{1}) \cdot g = D(f) \cdot g - D(\mathbb{1}) \cdot fg.$$

This then gives us a formula of $D(fg)$, written as

$$D(fg) = D(f) \cdot g + f \cdot D(g) - D(\mathbb{1}) \cdot fg.$$

On the other hand, the formula of $[D, f]$ also suggests to consider the map

$$D - D(\mathbb{1}) = [D, -](\mathbb{1})\colon \mathcal{O}(X) \longrightarrow \mathcal{O}(X), \quad f \longmapsto [D, f].$$

We claim that it is an element of $\mathcal{T}(X) = \operatorname{Der}(\mathcal{O}(X), \mathcal{O}(X))$. Indeed, since the k-linearity of $[D, -](\mathbb{1})$ in f is clear, it is clearly an endomorphism on $\mathcal{O}(X)$, and we only need to verify Leibniz rule. For this, we compute for any $f, g \in \mathcal{O}(X)$ that

$$\begin{aligned}
[D, fg](\mathbb{1}) &= D(fg) - D(\mathbb{1}) \cdot fg \\
&= D(f) \cdot g + f \cdot D(g) - 2D(\mathbb{1}) \cdot fg \\
&= (D(f) - D(\mathbb{1}) \cdot f) \cdot g + f \cdot (D(g) - D(\mathbb{1}) \cdot g) \\
&= [D, f](\mathbb{1}) \cdot g + f \cdot [D, g](\mathbb{1}).
\end{aligned}$$

Here the first and last equalities above are due to the identity $D - D(\mathbb{1}) = [D, -](\mathbb{1})$, and the second is by the prescribed formula of $D(fg)$. This finishes the proof of surjectivity because

$$D = D(\mathbb{1}) + [D, -],$$

with $D(\mathbb{1}) \in \mathcal{O}(X)$ and $[D, -] \in \mathcal{T}(X)$. $\square$

Problem 5.2 (Lecture 10, Exercise 7.3)**.** Let X be a smooth k-scheme dimension n. Prove: There is a unique right $\mathcal{D}_X$-module structure on Ω_X^n such that for local sections $f \in \mathcal{O}(U)$, $\partial \in \mathcal{T}(U)$, and $\omega \in \Omega_X^n(U)$, the right action is given by

$$\omega \cdot f = f\omega, \quad \omega \cdot \partial = -\mathcal{L}_\partial(\omega).$$

Solution. Since X is a smooth k-scheme, it suffices to check the unique $\mathcal{D}_X(U)$-module structure on $\Omega_X^n(U)$ for each open affine $U \subset X$. For this, we only need to concern about the action of $\mathsf{F}^{\leqslant 1}\mathcal{D}_X(U)$. By Problem 5.1, it suffices to consider the actions of $\mathcal{O}(U)$ and $\mathcal{T}(U)$.

Let $\{x_i\}_{i=1}^n$ be an étale coordinate system locally on U. Then, according to [Lecture 10, Proposition–Definition 1.21], $\{\mathrm{d}x_{i_1} \wedge \cdots \wedge \mathrm{d}x_{i_m} : 1 \leqslant i_1 < \cdots < i_m \leqslant n\}$ form a free basis of $\Omega_X^n(U)$ as an $\mathcal{O}(U)$-module, and $\{\partial_{i_1} \cdots \partial_{i_m} : 1 \leqslant i_1 < \cdots < i_m \leqslant n\}$ is the dual basis of $\mathcal{T}(U)$. Note that for $l \neq k$ we have $[\partial_l, \partial_k] = 0$. To recognize the action of $\mathcal{T}(U)$, we compute for $\omega \in \mathcal{O}_X^n(U)$ that

$$\omega \cdot (\partial_{i_1} \cdots \partial_{i_m}) = (-1)^m (\mathcal{L}_{\partial_{i_1}} \circ \cdots \circ \mathcal{L}_{\partial_{i_m}})(\omega).$$

Also, to check the compatibility of actions by $\mathcal{O}(U)$ and $\mathcal{T}(U)$, we need to check:

- The definition of Lie bracket $[\partial, f]$, i.e. $(\omega \cdot \partial) \cdot f - (f\omega) \cdot \partial = \omega \cdot \partial(f)$, and
- The composition equality $\omega \cdot (f\partial) = (\omega \cdot f) \cdot \partial$.

For the former, the given action means it is equivalent to $-f\mathcal{L}_\partial(\omega) + \mathcal{L}_\partial(f\omega) = \partial(f)\omega$. As for the latter, it suffices to check $-\mathcal{L}_{f\partial}(\omega) = -\mathcal{L}_\partial(f\omega)$. We see both equalities above are clear by definition of $\mathcal{L}_\partial$[14], so we get the right $\mathcal{D}_X$-module structure. The uniqueness also follows from the argument. □

Problem 5.3 (Lecture 10, Exercise 7.6)**.** In [Lecture 10, Example 7.5], prove $\mathcal{M}_{x^\lambda}$ is isomorphic to $\mathcal{O}_X$ as left $\mathcal{D}_X$-modules if and only if $\lambda \in \mathbb{Z}$.

Solution. Recall the construction of $\mathcal{M}_{x^\lambda}$ as follows. On $X = \mathbb{A}^1 - 0 = \operatorname{Spec} k[x^{\pm 1}]$, with $\lambda \in k$, the $\mathcal{T}_X$-action on the left $\mathcal{D}_X$-module $\mathcal{M}_{x^\lambda}$ is given by $\partial_x \cdot x^\lambda - \lambda x^{-1} \cdot x^\lambda = 0$, and then we have an $\mathcal{D}_X$-module isomorphism

$$\mathcal{M}_{x^\lambda} \simeq \mathcal{D}_X / \mathcal{D}_X \cdot (\partial_x - \lambda x^{-1}).$$

Note that we have an isomorphism between underlying $\mathcal{O}_X$-module structures; note that any isomorphism $\mathcal{M}_{x^\lambda} \simeq \mathcal{O}_X$ of $\mathcal{O}_X$-modules must be of form

$$\iota \colon \mathcal{O}_X \longrightarrow \mathcal{M}_{x^\lambda}, \quad f \longmapsto uf$$

for some $u \in \mathcal{O}_X^\times$. Fix such an ι in the following, then by PBW theorem for $\mathcal{D}_X$, this ι upgrades to an isomorphism of $\mathcal{D}_X$-modules if and only if u is such that

$$\partial_x(f) = \partial_x \cdot (uf),$$

where the right-hand side means the left action image of $\partial_x \in \mathcal{D}_X$. Since $\partial_x \in \mathcal{T}_X$, it satisfies Leibniz rule and we get

$$\partial_x \cdot (uf) = \partial_x(uf) - u \cdot \partial_x(f) = \partial_x(u) \cdot f = \lambda x^{-1} \cdot (uf).$$

As this holds for all $f \in \mathcal{O}_X$, we see $u = x^\lambda \in \mathcal{O}_X^\times$. Also, $x^\lambda \in \mathcal{O}_X^\times = k[x^{\pm 1}]^\times$ if and only if $\lambda \in \mathbb{Z}$. So we see ι is an isomorphism of $\mathcal{D}_X$-modules if and only if $\lambda \in \mathbb{Z}$. □

Problem 5.4 (Lecture 11, Exercise 6.7)**.** Let $x \colon \mathrm{pt} \to X$ be a closed point of X. We write $\delta_x \coloneqq x_{*,\mathrm{dR}}(k)$. Prove:

(1) $\delta_x \simeq k_x \otimes_{\mathcal{O}_X} \mathcal{D}_X$ as a right $\mathcal{D}_X$-module.
(2) δ_x is set-theoretically supported at x, i.e., for the complement open $U \coloneqq X - x$, we have $\delta_x|_U = 0$.
(3) There exists a unique section Dirac_x of δ_x such that $\mathsf{Dirac}_x \cdot f = f(x)\mathsf{Dirac}_x$ for any local section f of $\mathcal{O}_X$ defined near x.

[14] By definition, when U is affine with étale coordinate system $\{x_i\}_{i=1}^n$, if we write $\omega = \omega(x_1, \ldots, x_n)$, then

$$\mathcal{L}_\partial(\omega) \coloneqq \partial(\omega) - \sum_{i=1}^n \omega^{(i)},$$

where $\omega^{(i)} = \omega(x_1, \ldots, [\partial, x_i], \ldots, x_n)$.

(4) δ_x is generated by Dirac_x as a right $\mathcal{D}_X$-module.

Solution. (1) Note that x is an affine morphism (implying that $x_{*,\mathrm{dR}}$ is t-exact) and $\mathcal{D}_{\mathrm{pt}\to X}$ is locally free as an $\mathcal{O}_X$-module. Then by definition we have that

$$\delta_x = x_{*,\mathrm{dR}}(k) = x_*(k \otimes_{\mathcal{D}_{\mathrm{pt}}} \mathcal{D}_{\mathrm{pt}\to X}),$$

where $\mathcal{D}_{\mathrm{pt}} = \mathcal{O}_{\mathrm{pt}} = k$ and $\mathcal{D}_{\mathrm{pt}\to X} = x^*\mathcal{D}_X$. Combining this with derived projection formula, we get an isomorphism of $\mathcal{D}_X$-modules

$$\delta_x = x_*(k \otimes_{\mathcal{O}_{\mathrm{pt}}} x^*\mathcal{D}_X) \simeq x_*k \otimes_{\mathcal{O}_X} \mathcal{D}_X = k_x \otimes_{\mathcal{O}_X} \mathcal{D}_X.$$

(2) Consider the open embedding $j\colon U \hookrightarrow X$. Using the isomorphism of part (1), we have

$$j^*\delta_x \simeq j^*(k_x \otimes_{\mathcal{O}_X} \mathcal{D}_X) \simeq j^*k_x \otimes_{\mathcal{O}_X} j^*\mathcal{D}_X.$$

But as we know $k_x = x_*k$, it is clear that $j^*k_x = 0$. Here we regard k_x as the skyscraper sheaf at x and k the constant sheaf on X. So we have $\delta_x|_U = j^*\delta_x = 0$ as desired.

(3) We see from part (1) that $\delta_x \simeq k_x \otimes_{\mathcal{O}_X} \mathcal{D}_X$. We claim that $\mathsf{Dirac}_x \in \delta_x$ is the isomorphic image of the canonical element $1 \otimes \mathbb{1} \in k_x \otimes_{\mathcal{O}_X} \mathcal{D}_X$, and hence the existence and uniqueness follows. To check $1 \otimes \mathbb{1}$ satisfies the desired property, we let U_x be any open neighborhood of x in X and compute for any local section $f \in \mathcal{O}_X(U_x)$ that

$$(1 \otimes \mathbb{1}) \cdot f = 1 \otimes (\mathbb{1} \cdot f) = 1 \otimes f|_{\{x\}} = f(x)(1 \otimes \mathbb{1}).$$

This completes the proof of $\mathsf{Dirac}_x \cdot f = f(x)\mathsf{Dirac}_x$.

(4) By argument in part (3), we can identify Dirac_x with $1 \otimes \mathbb{1} \in k_x \otimes_{\mathcal{O}_X} \mathcal{D}_X$. Since $1 \otimes \mathbb{1}$ clearly generates $k_x \otimes_{\mathcal{O}_X} \mathcal{D}_X$ under the right $\mathcal{D}_X$-action, the conclusion follows. □

Problem 5.5 (Lecture 11, Exercise 8.3)**.** Let $x\colon \mathrm{pt} \to X$ be a closed point of X. Prove

$$\delta_x \otimes^! \delta_x \simeq \delta_x.$$

(A formal proof exists, but you are encouraged to do some direct calculations to see $\mathcal{H}^i(\delta_x \otimes_{\mathcal{O}_X} \delta_x) = 0$ unless $i = 0$, and $\mathcal{H}^0(\delta_x \otimes_{\mathcal{O}_X} \delta_x) = \delta_x$.)

Solution. We prove by direct calculations on cohomology. Using [Lecture 11, Construction 3.10], we have the formula

$$\delta_x \otimes^! \delta_x \simeq \delta_x \otimes_{\mathcal{O}_X} \delta_x \otimes_{\mathcal{O}_X} \omega_X^{-1}[-d_X].$$

Also, by Problem 5.4 we see $\delta_x \simeq k_x \otimes_{\mathcal{O}_X} \mathcal{D}_X$ is set-theoretically supported at x. This together with the formula above shows that $\delta_x \otimes_{\mathcal{O}_X} \delta_x$ is a quasi-coherent $\mathcal{D}_X$-module[15]. Now we construct a resolution for δ_x by constructing that for k_x. Taking an étale local coordinate system $\{x_i\}_{i=1}^{d_X}$ at x such that x corresponds to the origin, we get an étale map $U \to \mathbb{A}_k^{d_X}$ of schemes where U is an open neighborhood of x. This gives rise to the Koszul resolution $\mathsf{K}_\bullet$ of length d_X for the skyscraper sheaf k_x, written as

$$\mathsf{K}_\bullet = (K_{d_X} \to \cdots \to K_1 \to \mathcal{O}_X) \longrightarrow k_x \longrightarrow 0.$$

Note that each $K_i = \wedge^i K_1$ is a locally free (and hence projective) $\mathcal{O}_X$-module, and then each $K_i \otimes_{\mathcal{O}_X} \mathcal{D}_X$ is a locally projective $\mathcal{D}_X$-module. So $\mathsf{K}_\bullet \otimes_{\mathcal{O}_X} \mathcal{D}_X$ is the desired $\mathcal{D}_X$-module resolution of $k_x \otimes_{\mathcal{O}_X} \mathcal{D}_X \simeq \delta_x$, which is also locally projective of length d_X. Locally at x, if we write $\mathcal{H}^i(\delta_x \otimes_{\mathcal{O}_X} \delta_x) \coloneqq \mathcal{H}^i_{\mathrm{pt}}(X, \delta_x \otimes_{\mathcal{O}_X} \delta_x)$, then Serre duality dictates that

$$\mathcal{H}^i(\delta_x \otimes_{\mathcal{O}_X} \delta_x) \simeq \mathcal{E}xt^{d_X - i}_{\mathcal{O}_X}(\delta_x \otimes_{\mathcal{O}_X} \delta_x, \omega_X).$$

To compute the right-hand side, recall that the cohomology of any Koszul complex must vanish away from the last term. Thus, applying this property to $(\mathsf{K}_\bullet \otimes_{\mathcal{O}_X} \mathcal{D}_X) \otimes_{\mathcal{O}_X} (\mathsf{K}_\bullet \otimes_{\mathcal{O}_X} \mathcal{D}_X)$, we get

$$\mathcal{E}xt^j_{\mathcal{O}_X}(\delta_x \otimes_{\mathcal{O}_X} \delta_x, \omega_X) = \begin{cases} \delta_x \otimes_{\mathcal{O}_X} \delta_x, & \text{if } j = d_X, \\ 0, & \text{if } j \neq d_X. \end{cases}$$

[15]If this is true then $\delta_x \otimes_{\mathcal{O}_X} \delta_x$ will admit a resolution of length at most $2d_X$ by locally projective $\mathcal{D}_X$-modules.

Moreover, we have $\delta_x \otimes_{\mathcal{O}_X} \delta_x \simeq \delta_x$ because of the computation $(k_x \otimes_{\mathcal{O}_X} \mathcal{D}_X) \otimes_{\mathcal{O}_X} (k_x \otimes_{\mathcal{O}_X} \mathcal{D}_X) = (k_x \otimes_{\mathcal{O}_X} k_x) \otimes_{\mathcal{O}_X} (\mathcal{D}_X \otimes_{\mathcal{O}_X} \mathcal{D}_X) = k_x \otimes_{\mathcal{O}_X} \mathcal{D}_X$. Combining the ingredients above, it then follows that

$$\mathcal{H}^i(\delta_x \otimes_{\mathcal{O}_X} \delta_x) \simeq \begin{cases} \delta_x, & \text{if } i = 0, \\ 0, & \text{if } i \neq 0. \end{cases}$$

Therefore, for any $\mathcal{D}_X$-module $\mathscr{F}$,

$$\begin{aligned} \mathsf{Hom}_{\mathcal{D}_X}(\delta_x, \mathscr{F}) &\simeq \mathsf{Hom}_{\mathcal{D}_X}(\mathcal{H}^0(\delta_x \otimes_{\mathcal{O}_X} \delta_x), \mathscr{F}) \\ &\simeq \mathsf{Hom}_{\mathcal{D}_X}(\mathcal{E}xt^{d_X}_{\mathcal{O}_X}(\delta_x \otimes_{\mathcal{O}_X} \delta_x, \omega_X), \mathscr{F}) \\ &\simeq \mathsf{Hom}_{\mathcal{D}_X}(\mathcal{H}om_{\mathcal{O}_X}(\delta_x \otimes_{\mathcal{O}_X} \delta_x, \omega_X[d_X]), \mathscr{F}) \\ &\simeq \mathsf{Hom}_{\mathcal{D}_X}(\delta_x \otimes_{\mathcal{O}_X} \delta_x, \omega_X[d_X] \otimes_{\mathcal{O}_X} \mathscr{F}) \\ &\simeq \mathsf{Hom}_{\mathcal{D}_X}(\delta_x \otimes_{\mathcal{O}_X} \delta_x \otimes_{\mathcal{O}_X} \omega_X^{-1}[-d_X], \mathscr{F}) \\ &\simeq \mathsf{Hom}_{\mathcal{D}_X}(\delta_x \otimes^! \delta_x, \mathscr{F}). \end{aligned}$$

This finishes the proof of $\delta_x \otimes^! \delta_x \simeq \delta_x$ by Yoneda lemma. □

Alternative Solution. We also present a formal proof. Apply base change theorem to the following Cartesian diagram

$$\begin{tikzcd} \mathrm{pt} \arrow[r, "\mathrm{id}"] \arrow[d, "\mathrm{id}"] & \mathrm{pt} \arrow[d, "x"] \\ \mathrm{pt} \arrow[r, "x"] & X \end{tikzcd}$$

we see that $x^! \circ x_{*,\mathrm{dR}} \cong \mathrm{id}_{*,\mathrm{dR}} \circ \mathrm{id}^!$. Thus we are able to compute

$$\begin{aligned} \delta_x \otimes^! \delta_x &= x_{*,\mathrm{dR}} k \otimes^! \delta_x && \text{(by definition)} \\ &\simeq x_{*,\mathrm{dR}}(k \otimes^! x^! \delta_x) && \text{(by projection formula)} \\ &= x_{*,\mathrm{dR}}(k \otimes^! x^! x_{*,\mathrm{dR}} k) && \text{(by definition)} \\ &\simeq x_{*,\mathrm{dR}}(k \otimes^! \mathrm{id}_{*,\mathrm{dR}} \mathrm{id}^! k) && \text{(by base change)} \\ &\simeq x_{*,\mathrm{dR}}(k \otimes^! k). \end{aligned}$$

But it is clear that $k \otimes^! k \simeq k$, and hence we get $\delta_x \otimes^! \delta_x \simeq x_{*,\mathrm{dR}} k = \delta_x$. □

Homework 6

Problem 6.1 (Lecture 12, Exercise 3.9)**.** Show that

$$\dim \mathscr{F}\ell_G^{=w} = \ell(w).$$

If you do not know the basics about reductive groups, prove this for semisimple G.

Solution. By [Lecture 12, Lemma 3.7] we have that

$$\mathscr{F}\ell_G^{=w} \simeq N/(\mathrm{Ad}_w(N) \cap N) \simeq \mathrm{Ad}_w(N) \cap N^-.$$

Recall that $\dim N$ is the number of positive roots, and the goal now is to also reformulate $\mathrm{Ad}_w(N) \cap N^-$ in terms of positive roots in Φ^+. For this, provided the root space decomposition $\mathfrak{g} = \mathfrak{t} \oplus \bigoplus_{\alpha\in\Phi} \mathfrak{g}_\alpha$, let $X_\alpha \in \mathfrak{g}_\alpha$ be the root vector corresponding to α; this defines the root subgroup

$$U_\alpha := \{\exp(sX_\alpha) \colon s \in k\}.$$

Observe that each U_α is a 1-dimensional algebraic group over k, and we can view the generator of U_α as the map $u_\alpha \colon \mathbb{G}_\mathrm{a} \to G,\ s \mapsto \exp(sX_\alpha)$. Also, to consider the W-adjoint action on U_α, we compute $u_{w(\alpha)}(s) = \exp(sX_{w(\alpha)}) = \exp(s\,\mathrm{Ad}_w(X_\alpha)) = (\mathrm{Ad}_w(u_\alpha))(s)$, and it follows that

$$\mathrm{Ad}_w(U_\alpha) = U_{w(\alpha)}.$$

On the other hand, there are isomorphisms of k-schemes $N \simeq \prod_{\alpha\in\Phi^+} U_\alpha$ and $N^- \simeq \prod_{\alpha\in\Phi^-} U_\alpha$, so in particular $\mathrm{Ad}_w(N) \simeq \prod_{\alpha\in\Phi^+} U_{w(\alpha)}$. To conclude, we obtain

$$\mathrm{Ad}_w(N) \cap N^- \simeq \prod_{\alpha\in\Phi^+} U_{w(\alpha)} \cap \prod_{\beta\in\Phi^+} U_{-\beta} \simeq \prod_{\substack{\alpha\in\Phi^+\\ w(\alpha)\in\Phi^-}} U_\alpha,$$

because $U_{w(\alpha)} \cap U_{-\beta} = \emptyset$ unless $w(\alpha) = -\beta \in \Phi^-$. Again, by $\dim U_\alpha = 1$ we see

$$\dim \mathscr{F}\ell_G^{=w} = \#\{\alpha \in \Phi^+ \colon w(\alpha) \in \Phi^-\} = \ell(w),$$

where the last equality is by definition of $\ell(w)$. □

Problem 6.2 (Lecture 12, Exercise 4.9)**.** Deduce the BGG theorem from the localization theorem (see [Lecture 12, Theorem 4.7])[16].

Solution. Given $\lambda, \mu \in \mathfrak{t}^*$, there are w and w' such that $\lambda = w \cdot (-2\rho)$ and $\mu = w' \cdot (-2\rho)$. To deduce the BGG theorem, assume $[M_\lambda : L_\mu] \neq 0$ and it suffices to prove $w' \leq w$.

By localization theorem, we have

$$M_\lambda \longleftrightarrow \Delta_w, \quad L_\mu \longleftrightarrow \mathrm{IC}_{w'}.$$

Here $\Delta_w := i_{w,!}\mathcal{O}_{\mathscr{F}\ell_G^{=w}}$ with $i_w \colon \mathscr{F}\ell_G^{=w} \hookrightarrow \mathscr{F}\ell_G$. Since the Schubert cell $\mathscr{F}\ell_G^{=w}$ is a locally closed subset of $\mathscr{F}\ell_G$ and $\mathrm{supp}(\Delta_w) \subset \mathrm{supp}(\mathcal{O}_{\mathscr{F}\ell_G^{=w}})$ by property of !-pushforward, we see Δ_w is set-theoretically supported on the closure of $\mathscr{F}\ell_G^{=w}$, which is $\mathscr{F}\ell_G^{\leq w}$. In particular, any subquotient of Δ_w is set-theoretically supported on $\mathscr{F}\ell_G^{\leq w}$. From the assumption $[M_\lambda : L_\mu] \neq 0$, we know $\mathrm{IC}_{w'}$ is isomorphic to a subquotient of Δ_w, and hence supported on $\mathscr{F}\ell_G^{\leq w}$.

Provided the above, we get $\mathrm{supp}(\Delta_{w'}) = \mathscr{F}\ell_G^{\leq w'}$, and it suffices to show the Bruhat cell $C_{w'} := \mathscr{F}\ell_G^{=w'}$ is contained in $\mathrm{supp}(\mathrm{IC}_{w'})$. Indeed, by definition we have $\mathrm{IC}_{w'} = \mathrm{im}(\Delta_{w'} \to \nabla_{w'})$; on the other hand $\Delta_{w'}|_{C_{w'}} = \nabla_{w'}|_{C_{w'}} = \mathcal{O}_{C_{w'}}$, implying that $\mathrm{IC}_{w'}|_{C_{w'}} = \mathcal{O}_{C_{w'}}$. This proves that $\mathrm{IC}_{w'}$ is non-zero when restricted to $\mathscr{F}\ell_G^{=w'}$, and hence $w' \leq w$. It follows that $\mu \preceq_\subset \lambda$ as desired. □

Problem 6.3 (Lecture 12, Exercise 4.10)**.** Prove that when $G = \mathrm{SL}_2$, the homomorphism $\alpha \colon U(\mathfrak{g}) \to \mathcal{D}(\mathscr{F}\ell_G)$ induces an isomorphism

$$U(\mathfrak{g})_{\chi_0} \simeq \mathcal{D}(\mathscr{F}\ell_G).$$

This is a special case of part (1) of localization theorem (see [Lecture 12, Theorem 4.7]).

[16] Hint: Prove any subquotient of Δ_w is set-theoretically support on the Schubert variety $\mathscr{F}\ell_G^{\leq w}$.

Solution. When $G = \mathrm{SL}_2$, we have $\mathscr{F}\ell_G = \mathbb{P}^1$ and need to consider $\alpha\colon U(\mathfrak{g}) = U(\mathfrak{sl}_2) \longrightarrow \mathcal{D}(\mathbb{P}^1)$. Let e, f, h be a standard basis of $\mathfrak{g} = \mathfrak{sl}_2$. Recall that $Z(U(\mathfrak{g}))$ is generated by the Casimir element $\Omega = ef + fe + h^2/2 \in U(\mathfrak{g})$. Let $U = \{(x : 1)\colon x \in k\}$ be a standard affine open in $\mathbb{P}^1$. For each coordinate parameter u of U, by computing

$$\alpha(\mathbf{g}) \cdot u = \frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=0} \exp(t\mathbf{g})(\exp(-t\mathbf{g})u), \quad \mathbf{g} \in \{e, f, h\},$$

we see, depending on the choice of u,

$$\alpha(e) = -\partial_u, \quad \alpha(f) = u^2\partial_u, \quad \alpha(h) = -2u\partial_u.$$

It follows that

$$\begin{aligned}\alpha(\Omega) &= \alpha(ef + fe + h^2/2)\\ &= (-\partial_u)u^2\partial_u + u^2\partial_u(-\partial_u) + (2u\partial_u)^2/2\\ &= -2u\partial_u - u^2\partial_u^2 - u^2\partial_u^2 + 2u\partial_u + 2u^2\partial_u^2\\ &= 0.\end{aligned}$$

Since the central character χ_0 corresponds to $Z(U(\mathfrak{g})) = k[\Omega]$ and $\alpha(\Omega) = 0$, the map α factors through $U(\mathfrak{g})_{\chi_0}$. So we get

$$\alpha_{\chi_0}\colon U(\mathfrak{g})_{\chi_0} \longrightarrow \mathcal{D}(\mathscr{F}\ell_G).$$

Note that both sides of α_{χ_0} admits graded structure, and we claim that

$$\mathrm{gr}^1(U(\mathfrak{g})_{\chi_0}) = \mathfrak{g} \twoheadrightarrow \mathcal{T}(\mathscr{F}\ell_G) = \mathrm{gr}^1\mathcal{D}(\mathscr{F}\ell_G)$$

is surjective. If this is true, then α_{χ_0} is surjective as well. Indeed, the claim follows from that $\mathcal{T}(\mathscr{F}\ell_G) = \mathcal{O}_{\mathscr{F}\ell_G}(2)$, whose global section is generated by $\alpha(e), \alpha(f), \alpha(h)$, meaning that α is surjective when restricted to $\mathfrak{g}$. Thus we have proved the claim as well as the surjectivity of α_{χ_0}. Now it remains to check the injectivity. For this, we have the embedding $\mathrm{gr}^\bullet\mathcal{D}(\mathscr{F}\ell_G) \hookrightarrow (\mathrm{Sym}^\bullet \mathcal{T})(\mathscr{F}\ell_G)$ that is an identity on degree 1 part. Also, the diagram below commutes:

$$\begin{tikzcd}
\mathrm{gr}^\bullet(U(\mathfrak{g})_{\chi_0}) \arrow[r, "\alpha_{\chi_0}"] \arrow[dr, two heads] & \mathrm{gr}^\bullet\mathcal{D}(\mathscr{F}\ell_G) \arrow[d, hook] \\
& (\mathrm{Sym}^\bullet \mathcal{T})(\mathscr{F}\ell_G).
\end{tikzcd}$$

Here the map $\mathrm{gr}^\bullet(U(\mathfrak{g})_{\chi_0}) \twoheadrightarrow (\mathrm{Sym}^\bullet \mathcal{T})(\mathscr{F}\ell_G)$ is surjective because we have shown α_{χ_0} is surjective on degree 1 part. To show this map is injective, we only need for all $n \geqslant 1$ that

$$\dim \mathrm{gr}^n(U(\mathfrak{g})_{\chi_0}) = \dim (\mathrm{Sym}^n \mathcal{T})(\mathscr{F}\ell_G).$$

Note that $\mathrm{Sym}^n \mathcal{T} = \mathrm{Sym}^n \mathcal{O}(2) = \mathcal{O}_{\mathbb{P}^1}(2n)$[17], so the right-hand side equals $\dim \Gamma(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(2n)) = 2n+1$. To compute the left-hand side, using PBW theorem, $\mathrm{gr}^n(U(\mathfrak{g})_{\chi_0})$ has a basis $\{e^i f^{n-i}\colon 0 \leqslant i \leqslant n\} \cup \{e^i f^{n-i-1}h\colon 0 \leqslant i \leqslant n-1\}$, so it also has dimension $2n+1$. This completes the proof that α_{χ_0} is injective, and hence an isomorphism. □

Problem 6.4 (Lecture 13, Exercise 1.5)**.** Let $e \in X \simeq G/B$ be the closed point corresponding to the chosen Borel subgroup B. We aim to prove that, as stated in the localization theorem, we can produce the (left) Verma module $M_{-2\rho}$ with highest weight -2ρ if using the left $\mathcal{D}$-module corresponding to δ_e. Let $\delta_e^{\mathrm{l}} \simeq \delta_e \otimes \omega_X^{-1}$ be the left $\mathcal{D}$-module corresponding to δ_e. Consider the left $U(\mathfrak{g})$-module $V \coloneqq \Gamma(X, \delta_e^{\mathrm{l}})$.

(1) Prove: There is a *canonical* isomorphism

$$\delta_e^{\mathrm{l}} \simeq \mathcal{D}_X \otimes_{\mathcal{O}_X} \ell,$$

where ℓ is the fiber of ω_X^{-1} at e, viewed as a skyscraper sheaf.

(2) Let $\ell \hookrightarrow V$ be the injection induced by taking global sections for the embedding $\mathcal{O}_X \otimes_{\mathcal{O}_X} \ell \hookrightarrow \mathcal{D}_X \otimes_{\mathcal{O}_X} \ell$. Prove[18]: This line in V is a weight subspace of weight -2ρ.

[17] See the proof of Problem 6.5 for details in this standard fact.

[18] Hint: $\ell \simeq \wedge^d \mathcal{T}_{X,e}$ and $\mathcal{T}_{X,e} \simeq \mathfrak{n}^-$.

(3) Prove[19]: The subalgebra $\mathfrak{b} \subset \mathfrak{g}$ stabilizes the line $\ell \subset V^{2\rho}$.
(4) Construct a $U(\mathfrak{g})$-linear map
$$M_{-2\rho} \longrightarrow V$$
and prove it is an isomorphism.

Solution. (1) By Problem 5.4, there is an isomorphism $\delta_e \simeq k_e \otimes_{\mathcal{O}_X} \mathcal{D}_X$ of right $\mathcal{D}_X$-modules. On the other hand, by definition we have the commutative diagram

$$\begin{tikzcd} D(\mathcal{O}_X\text{-}\mathsf{mod}_{\mathrm{qc}}) \arrow[r, "(-)\otimes_{\mathcal{O}_X}\omega_X^{-1}"] \arrow[d, "\mathrm{ind}^{\mathrm{r}}"'] & D(\mathcal{O}_X\text{-}\mathsf{mod}_{\mathrm{qc}}) \arrow[d, "\mathrm{ind}^{\mathrm{l}}"] \\ D(\mathcal{O}_X\text{-}\mathsf{mod}_{\mathrm{qc}}) \arrow[r, "(-)\otimes_{\mathcal{O}_X}\omega_X^{-1}"] & D(\mathcal{O}_X\text{-}\mathsf{mod}_{\mathrm{qc}}) \end{tikzcd}$$

which leads to $k_e \otimes_{\mathcal{O}_X} \mathcal{D}_X \otimes_{\mathcal{O}_X} \omega_X^{-1} \simeq \mathcal{D}_X \otimes_{\mathcal{O}_X} k_e \otimes_{\mathcal{O}_X} \omega_X^{-1}$. If we write $i\colon e \hookrightarrow X$, then by construction we have an isomorphism $\ell \simeq i_* i^* \omega_X^{-1}$ of $\mathcal{O}_X$-modules. Thus,

$$\begin{aligned} \delta_e^{\mathrm{l}} &\simeq \mathcal{D}_X \otimes_{\mathcal{O}_X} i_* k \otimes_{\mathcal{O}_X} \omega_X^{-1} && \text{(by argument above)} \\ &\simeq \mathcal{D}_X \otimes_{\mathcal{O}_X} i_*(k \otimes_{\mathcal{O}_e} i^*\omega_X^{-1}) && \text{(by projection formula)} \\ &= \mathcal{D}_X \otimes_{\mathcal{O}_X} i_* i^* \omega_X^{-1} && \text{(by } \mathcal{O}_e = k\text{)} \\ &\simeq \mathcal{D}_X \otimes_{\mathcal{O}_X} \ell. && \text{(by prescribed description of } \ell\text{)} \end{aligned}$$

(2) By construction we have $\ell \simeq \wedge^d \mathcal{T}_{X,e} \simeq \wedge^d \mathfrak{n}^-$. Let v be a generator of this line ℓ, then v must be of form $\bigwedge_{\alpha\in\Phi^-} v_\alpha$. For each $h \in \mathfrak{h}$, we have by definition of $\mathfrak{h}$-action that
$$h(v) = \sum_{\beta\in\Phi^-} \bigwedge_{\alpha\in\Phi^- - \{\beta\}} v_\alpha \wedge h(v_\beta) = \sum_{\beta\in\Phi^-} \beta(h) \cdot \bigwedge_{\alpha\in\Phi^-} v_\alpha = \sum_{\beta\in\Phi^-} \beta(h)\cdot v.$$
Note that $\sum_{\beta\in\Phi^-} \beta(h) = -2\rho(h)$, so $\ell \subset V$ is a weight subspace of weight -2ρ.

(3) Recall that the PBW theorem for $\mathcal{D}_X$ dictates $\mathrm{gr}^\bullet \mathcal{D}_X \cong \mathrm{Sym}^\bullet_{\mathcal{O}_X} \mathcal{T}_X$, which further gives rise to a canonical filtration on V, where $\mathsf{F}^{\leqslant i}V$ is the image of ℓ under the action of $\mathsf{F}^{\leqslant i}\mathcal{D}_X$. Recall that the $\mathfrak{g}$-action is realized by $\mathcal{D}_X$ through the map $\mathfrak{g} \to \mathcal{T}_X \to \mathsf{F}^{\leqslant 1}\mathcal{D}_X \hookrightarrow \mathcal{D}_X$. Restricting this to $\mathfrak{b} \subset \mathfrak{g}$, the map $\mathfrak{b} \otimes \ell \to V$ factors through $\mathsf{F}^{\leqslant 1}V$ (as the canonical map $U(\mathfrak{g}) \to \mathcal{D}_X$ is compatible with the filtration). To show that ℓ is stabilized by $\mathfrak{b}$, it suffices to show that via the prescribed map $\mathfrak{b} \hookrightarrow \mathfrak{g} \to \mathcal{D}_X$, the image of $\mathfrak{b} \otimes \ell$ lies in $\mathcal{O}_X \otimes_{\mathcal{O}_X} \ell$. For this, we only need to show the composition
$$\mathfrak{b} \otimes \ell \longrightarrow \mathsf{F}^{\leqslant 1}V \longrightarrow \mathrm{gr}^1 V$$
is zero; but we observe $\mathrm{gr}^1 V = (\mathrm{gr}^1 \mathcal{D}_X)(\ell) = \mathcal{T}_X(\ell) \simeq \mathfrak{n}^- \otimes \ell$, which means the desired composition is induced by the map $\mathfrak{b} \to \mathfrak{n}^-$.

(4) Choose a set-theoretical map $M_{-2\rho} \to V$ such that the highest weight vector $v_{-2\rho} \in M_{-2\rho}$ is mapped to a generator $v \in \ell$ of $\ell \subset V$. By part (3), $\ell = kv$ is stabilized by $\mathfrak{b}$, so the map
$$M_{-2\rho} = k_{-2\rho} \otimes_{U(\mathfrak{b})} U(\mathfrak{g}) \longrightarrow V$$
is $U(\mathfrak{g})$-linear. To show this is an isomorphism, it suffices to show the bijectivity. The surjectivity follows from that V is generated by $\mathcal{D}_X$ from ℓ and $U(\mathfrak{g}) \twoheadrightarrow \mathcal{D}_X$ is surjective. As for the injectivity, suppose $t \in U(\mathfrak{n}^-)$ is such that $t \cdot v = 0$ in V, then we must have $t \cdot v_{-2\rho} = 0$ in $M_{-2\rho}$ because the map is $U(\mathfrak{g})$-linear. This completes the proof that $M_{-2\rho} \simeq V$. □

Problem 6.5 (Lecture 13, Exercise 3.12)**.** For $G = \mathrm{SL}_2$, prove $\mathfrak{p}\colon \widetilde{\mathcal{N}} \to \mathcal{N}$ is the blow-up of $\mathcal{N}$ at the point $0 \in \mathcal{N}$.

[19]Hint: Consider the PBW filtration of $\mathcal{D}_X$ and the induced filtration on V. Show that $\mathfrak{b} \otimes \ell \to V$ factors through $\mathsf{F}^{\leqslant 1}V$ and the composition $\mathfrak{b} \otimes \ell \to \mathsf{F}^{\leqslant 1}V \to \mathrm{gr}^1 V$ is zero.

Solution. For $\mathfrak{g} = \mathfrak{sl}_2$, we have $\mathcal{N} = \{X \in \mathfrak{sl}_2 \colon \det X = 0\}$. If we write $X = \left(\begin{smallmatrix} a & b \\ c & -a \end{smallmatrix}\right) \in \mathrm{M}_2(k)$ satisfying the condition $\det X = -a^2 - bc = 0$, we can realize the nilpotent cone $\mathcal{N}$ as the affine k-scheme

$$\mathcal{N} = \operatorname{Spec} k[a, b, c]/(a^2 + bc).$$

Recall that the blow-up of $\mathcal{N}$ at 0 is $\operatorname{Proj}(\mathcal{R}(\mathcal{I}))$, where $\mathcal{R}(\mathcal{I}) = \bigoplus_{n \geqslant 0} \mathcal{I}^n$ is the Rees algebra with $\mathcal{I}$ the defining ideal of $0 \in \mathcal{N}$, generated by images of a, b, c satisfying $a^2 + bc = 0$.

On the other hand, by definition of Springer resolution, at $0 \in \mathcal{N}$ we have

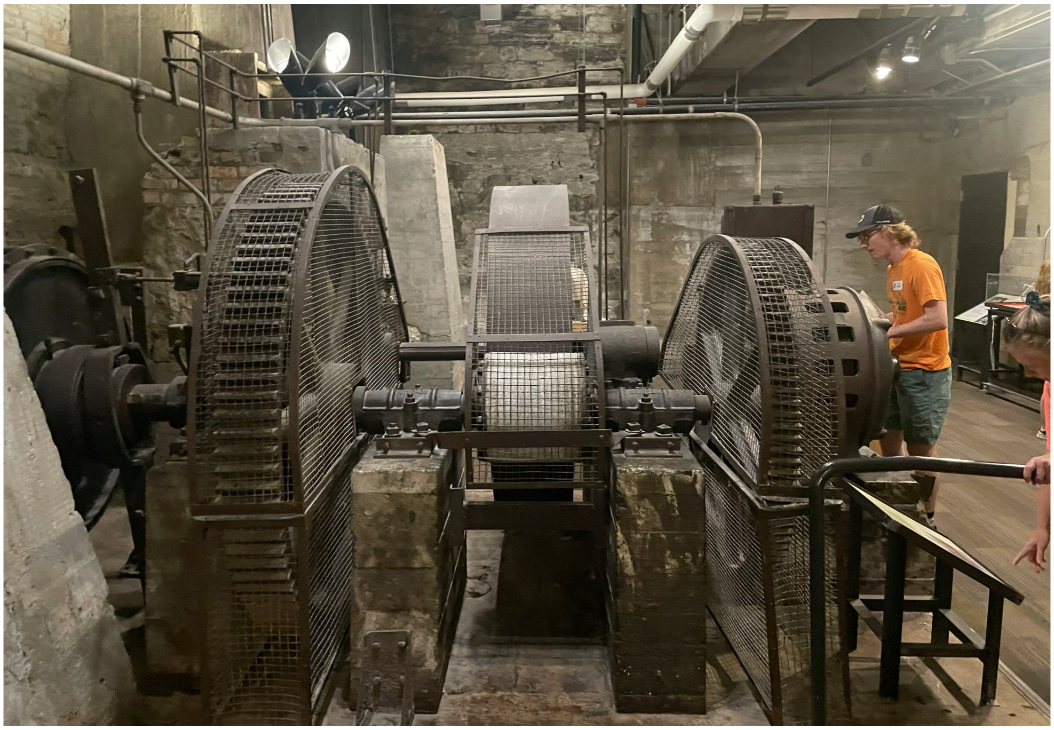
$$\widetilde{\mathcal{N}} \simeq T^*(G/B) = T^*\mathbb{P}^1 \simeq \operatorname{Spec}_{\mathbb{P}^1}(\operatorname{Sym}^\bullet_{\mathcal{O}_{\mathbb{P}^1}} \mathcal{T}_{\mathbb{P}^1}).$$

Here the last isomorphism is given by [Lecture 13, Construction 2.4]. To compute the relative spectrum on the right-hand side, we use $\mathcal{T}_{\mathbb{P}^1} = \mathcal{O}_{\mathbb{P}^1}(2)$[20] to get

$$\operatorname{Sym}^\bullet_{\mathcal{O}_{\mathbb{P}^1}} \mathcal{T}_{\mathbb{P}^1} = \operatorname{Sym}^\bullet_{\mathcal{O}_{\mathbb{P}^1}} \mathcal{O}_{\mathbb{P}^1}(2) = \bigoplus_{n \geqslant 0} \operatorname{Sym}^n \mathcal{O}_{\mathbb{P}^1}(2) = \bigoplus_{n \geqslant 0} \mathcal{O}_{\mathbb{P}^1}(2n).$$

Combining these up, to show that $\widetilde{N}$ is the blow-up of $\mathcal{N}$ at 0, it remains to check $\operatorname{Proj}(\mathcal{R}(\mathcal{I})) \simeq \operatorname{Spec}_{\mathbb{P}^1}(\bigoplus_{n \geqslant 0} \mathcal{O}_{\mathbb{P}^1}(2n))$ as k-schemes. But this is true because each section of $\mathcal{R}(\mathcal{I})$ generates a regular function in a, b, c, and the degree of a is reduced by $2k \in \mathbb{Z}$ for some k via the relation $a^2 + bc = 0$. $\square$

[20]For an explanation, notice that $\deg(\mathcal{T}_{\mathbb{P}^1}) + \deg(\mathcal{O}_{\mathbb{P}^1}) = \chi(\mathbb{P}^1) = 2$ with $\deg(\mathcal{O}_{\mathbb{P}^1}) = 0$, so $\mathcal{T}_{\mathbb{P}^1} = \mathcal{O}_{\mathbb{P}^1}(2)$. Alternatively, we can construct the tangent bundle $\mathcal{T}_{\mathbb{P}^1}$ explicitly as follows. Cover $\mathbb{P}^1$ by standard affine opens $U_0 = \{(1 : x_1) \colon x_1 \in k\}$ and $U_1 = \{(x_0 : 1) \colon x_0 \in k\}$ and write down the transition function on $U_0 \cap U_1$ as $x_1 = x_0^{-1}$. It follows that $d(x_0^{-1})/dx = -1/x^2$, which also proves $\mathcal{T}_{\mathbb{P}^1} = \mathcal{O}_{\mathbb{P}^1}(2)$.



Photograph — June 8, 2024; at the Mill City Museum, Minneapolis, Minnesota. *The prototype of a machine of flour manufacturing industry over which the city was once upon celebrated.*

References

[Gai05] Dennis Gaitsgory. *Course Notes for Geometric Representation Theory*. 2005. available at https://people.mpim-bonn.mpg.de/gaitsgde/267y/cat0.pdf.

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