Lecture 1: Introduction and Preliminaries

<u>Reference</u>: Ben Green's notes.

<u>Tentative Plan</u>: (1) Advanced Fourier analysis ← especially for students in the 3+X Program.

(2) Additive number theory (covering Goldbach Conjecture).

(3) Additive Combinatorics.

§1 Highlights of the Course

Ⓐ <u>Thm</u> (Lagrange) Every positive integer can be written as the sum of 4 squares of integers: $n = n_1^2 + n_2^2 + n_3^2 + n_4^2$.

Ⓑ <u>Thm</u> (Waring's Problem; Hilbert & Hardy–Littlewood).

For every $k \geq 2$, $\exists\, s$ s.t. $\forall n \in \mathbb{Z}$, $\exists\, n_1, \cdots, n_s \in \mathbb{Z}_{\geq 0}$,

$$n = \sum_{i=1}^{s} n_i^k$$

(s kth powers of nonnegative ints).

Ⓒ <u>Thm</u> (Roth) Let $A \subseteq \mathbb{N}$ have positive upper density, i.e. $\limsup_{N \to \infty} \frac{|A \cap [1,N]|}{N} > 0$.

Then A contains infinitely many nontrivial 3-term arithmetic progressions.

Ⓓ <u>Thm</u> (Freiman) Let $A \subseteq \mathbb{Z}$ be a set of <u>small doubling</u> (that is, $|A+A| \leq K|A|$ for some $K \geq 1$.)

Then A is contained in a generalized arithmetic progression P such that $\dim |P|$ & density $|P|/|A|$ are both bounded in terms of K.

§2 Landau and Vinogradov Notation

The following notation will be used extensively.

- $A = O(B)$ if $\exists$ const $C > 0$ s.t. $|A| \le C|B|$.
- Usually, $A = A(x)$, $B = B(x)$. (And we're interested in $x \to \infty$.)

E.g. $P(x) = O(Q(x))$ whenever $\deg P < \deg Q$, $\sin x = O(1)$ $(x \to \infty)$.

Note: We also denote $A = O(B)$ by $A \ll B$ or $B \gg A$.

E.g. $x^{1-\varepsilon} \ll_{\varepsilon} \frac{x}{\log x} \ll x \ll \frac{x}{100} - 100 \ll x \log x \ll x^2$ $(x \to \infty)$.

$C = C(\varepsilon)$

We denote $A = o(B)$ if $B(x)/A(x) \to 0$ as $x \to \infty$

this differs from the notn in Calculus.

E.g. $1/\log x = o(1)$, $\log x = x^{o(1)}$ $(x \to \infty)$

§3 Preliminaries: Fourier Transform

We will need the Fourier transform on 3 groups ($\mathbb{Z}/q\mathbb{Z}$, $\mathbb{Z}$, $\mathbb{R}$). Basic knowledge of what these are and how they work is essential (but we don't need their finer properties).

Ⓐ Fourier transforms

(1) If $f: \mathbb{Z}/q\mathbb{Z} \to \mathbb{C}$, define

$$\hat{f}(r) := \sum_{x \in \mathbb{Z}/q\mathbb{Z}} f(x) e\left(-\frac{rx}{\boxed{q}}\right), \qquad e(x) := e^{2\pi i x}.$$

(2) If $f: \mathbb{Z} \to \mathbb{C}$ is "nice", define

$$\hat{f}(\theta) := \sum_{n \in \mathbb{Z}} f(n) e(-n\theta).$$

(3) If $f: \mathbb{R} \to \mathbb{C}$ is "nice", define

$$\hat{f}(\xi) := \int_{\mathbb{R}} f(x) e(-\xi x)\, dx.$$

Ⓑ Parseval's formula

(This is perhaps the most important property for us).

(1) If $f : \mathbb{Z}/q\mathbb{Z} \longrightarrow \mathbb{C}$, we have

$$\sum_{r \in \mathbb{Z}/q\mathbb{Z}} |\hat{f}(r)|^2 = \boxed{\frac{1}{q}} \sum_{x \in \mathbb{Z}/q\mathbb{Z}} |f(x)|^2 .$$

(2) If $f : \mathbb{Z} \to \mathbb{C}$ is "nice", we have

$$\int_0^1 |\hat{f}(\theta)|^2 \, d\theta = \sum_{n \in \mathbb{Z}} |f(n)|^2 .$$

(3) If $f : \mathbb{R} \to \mathbb{C}$ is "nice", we have

$$\int_{\mathbb{R}} |\hat{f}(\xi)|^2 \, d\xi = \int_{\mathbb{R}} |f(x)|^2 \, dx .$$

$\Big\}$ The proofs are straightforward.