Lecture 3: Waring's Problem and the Circle Method

§1 Waring's Problem

Recall Lagrange's thm: $\forall\, n \geq 1$, $n = a^2 + b^2 + c^2 + d^2$.

Waring (1770) conjectured:

Waring's Problem Let $k \geq 2$. Then $\exists\, s = s(k)$ s.t. $\forall\, n \in \mathbb{N}$,

$$n = \sum_{i=1}^{s} x_i^k, \quad x_i \in \mathbb{Z}_{\geq 0}.$$

Finally proved by Hilbert (1909) & Hardy-Littlewood (1920s).

It suffices to prove that every $\boxed{\text{large}}$ n is a sum of kth powers.

Def'n Denote by $G(k)$ be the least integer s s.t. $\forall\, n \gg 0$,

$$\exists\, x_1, \cdots, x_s \in \mathbb{Z}_{\geq 0} \text{ s.t. } \sum_{i=1}^{s} x_i^k = n.$$

Big Theorem (Solution to Waring's Problem)

We obtain $G(k) < 100^k < \infty$.

(Morally, Hua's lemma $\Rightarrow G(k) < 2^k + 1$.)

$\begin{cases} \text{Best lower bound: easy, } G(k) \geq k+1. \\ \text{Best upper bound: deep result by Wooley:} \end{cases}$

$$G(k) \leq k \log k + k \log(\log k) + O(k).$$

- $G(2) = 4$, $G(4) = 16$, BUT all other values unknown!
- Open problem Is $G(3) = 4$?

We will follow (a modern version of) Hardy and Littlewood's proof.
This is based on their influential circle method.

§2 Asymptotic Formula

We will prove an asymptotic formula.

Thm (Asymptotics for Waring's problem).

Let $r_{k,s}(N) := \#$ rep'ns of N as $x_1^k + \cdots + x_s^k$ $(x_i \geq 0)$.

Suppose that $s \geq 100^k$. Then

$$r_{k,s}(N) = \mathfrak{S}_{k,s}(N) \cdot N^{\frac{s}{k}-1} + o(N^{\frac{s}{k}-1})$$

where the singular series is

$$\mathfrak{S}_{k,s}(N) = \beta_\infty \prod_p \beta_p(N).$$

and $\beta_p(N) =$ local density of solutions,

$$\beta_p(N) := \lim_{n\to\infty} p^{-n(s-1)} \left| \{(x_1, \cdots, x_s) \in (\mathbb{Z}/p^n\mathbb{Z})^s : \sum_{i\leq s} x_i^k \equiv N (p^n)\} \right|.$$

and the Archimedean density is

$$\beta_\infty := \Gamma(1+\tfrac{1}{k})^s / \Gamma(\tfrac{s}{k}).$$

The asymptotic formula is complemented by

Thm (Singular series)

For $s \geq k^4$ we have $1 \ll \mathfrak{S}_{k,s}(N) \ll 1$ (i.e. $\mathfrak{S}_{k,s} \asymp 1$).

Interpretation of the asymptotic formula.

It implies that $r_{k,s}(N) \asymp N^{\frac{s}{k}-1}$.

This is the expected order of magnitude, since we can show by elementary means that

$$c_{s,k} N^{\frac{s}{k}} \leq \sum_{n\leq N} r_{k,s}(n) \leq C_{s,k} N^{\frac{s}{k}}.$$

The asymptotic formula is a local-to-global principle:

if $P(x_1, \cdots, x_s) = x_1^k + \cdots + x_s^k$, then

$$|\{\bar{x} \in \mathbb{N}_0^s : P(\bar{x}) = N\}| \sim \beta_\infty N^{\frac{s}{k}-1} \prod_p \lim_{n\to\infty} \Pr(\bar{x} \in (\mathbb{Z}/p^n\mathbb{Z})^s : P(\bar{x}) \equiv N \bmod p^n).$$

and one can show that $\beta_\infty N^{\frac{s}{k}-1} = \text{Area}(\bar{x} \in \mathbb{R}_{\geq 0} : P(\bar{x}) = N)$.

§3 The Hardy-Littlewood Circle Method

(This method will underlay what we do in the next few lectures. The details are somewhat complicated, but the IDEAS are important.)

Very roughly, the circle method tells us that

Counting sol'ns to $a_1 + \cdots + a_s = N$, $a_i \in A$ $\longleftrightarrow$ Estimating $\hat{1}_A(\theta)$, $\forall \theta \in \mathbb{T}$.

However, need $\boxed{s \gg 0}$ $\leftarrow$ depending on the problem at hand

Key Object Fourier transform (or exponential sum)

$$\hat{1}_A(\theta) = \sum_{n \in A} e(-\theta n), \quad e(x) = e^{2\pi i x}.$$

More precisely:

Big Theorem If $A \subseteq \mathbb{Z}$ finite, then

$$|\{(a_1, \cdots, a_s) \in A^s : a_1 + \cdots + a_s = N\}| = \int_0^1 \boxed{\hat{1}_A(\theta)^s} e(N\theta)\, d\theta.$$

Proof $\text{LHS} = \sum_{a_1, \cdots, a_s \in A} 1_{a_1 + \cdots + a_s - N = 0}(a_1, \cdots, a_s)$

Use $1_{n=0}(n) = \int_0^1 e(nx)\,dx$ and change order of integral and sum. □

Corollary Denoting $X = \{n^k : n \leq N^{1/k}\}$,

$$r_{k,s}(N) = \int_0^1 \hat{1}_X(\theta)^s e(N\theta)\, d\theta.$$

We now need to estimate $\hat{1}_X(\theta)$ uniformly in θ.

Heuristic Typically, $\hat{1}_X(\theta)$ large when $\theta \approx a/q$, a, q "small"

$\hat{1}_X(\theta)$ small when θ highly irrational.

Example Suppose $\theta = 1/3$, $k=2$. Then

$$\hat{1}_X(\theta) = \sum_{n \le N^{1/2}} e\left(\frac{n^2}{3}\right) = \left(\frac{1}{3} + \frac{2}{3}e\left(\frac{1}{3}\right) + o(1)\right) N^{\frac{1}{2}} \gg N^{\frac{1}{2}}.$$

Similarly, if $\theta = a/q$, $\underbrace{q = O(1)}_{\text{namely, } q \text{ is ``small''}}$, then

$$\hat{1}_X(\theta) = \sum_{n \le N^{1/2}} e(an^2/q) \gg N^{\frac{1}{2}}.$$

⊛ Lastly, if we perturb θ by c/N for small $c>0$, nothing changes.

Example Let $\theta = \sqrt{2}$, $k=2$. Then

$$\hat{1}_X(\theta) = \sum_{n \le N^{1/2}} e(\sqrt{2}n^2).$$

Expect: the sequence $\sqrt{2}n^2 \bmod 1$ to be equidistributed in $[0,1]$.
⇝ and therefore, expect $\hat{1}_X(\theta) = o(N^{1/2})$.
Again, the same holds if we perturb θ by $o(\frac{1}{N})$.

Let $\|\theta\| :=$ the dist from θ to the nearest integer.
Also identify $\mathbb{T} = \mathbb{R}/\mathbb{Z} \approx [0,1)$.

To talk rigorously about rationals of small denominator:

Def'n Set $\eta := \frac{1}{10k}$. Define the major arcs

$$\mathfrak{M} = \bigcup_{\substack{q \le N^{\eta} \\ a \in (\mathbb{Z}/q\mathbb{Z})^*}} \mathfrak{M}_{a,q} \qquad \text{(for fixed } N\text{)},$$

where

the arc length (not the Euclidean length).

$$\mathfrak{M}_{a,q} := \left\{\theta \in \mathbb{T} : \boxed{\left|\theta - \frac{a}{q}\right|} \le N^{-1+2\eta}\right\}.$$

Define the minor arcs $\mathfrak{m} := \mathbb{T} \setminus \mathfrak{M}$.

Roughly: $\theta \in \mathfrak{M}$ if $\exists\, q \lesssim 1$ s.t. $\|q\theta\| \lesssim \frac{1}{N}$.

<u>Upshot</u> The idea of the circle method is to evaluate

$$\int_{\mathfrak{M}} \widehat{1_X}(\theta)^s e(N\theta)\, d\theta, \qquad \int_{\mathfrak{m}} \widehat{1_X}(\theta)^s e(N\theta)\, d\theta.$$

Why is this called the circle method?

(Think of: $\mathbb{T}$ = unit circle of $\mathbb{C}$ via $x \mapsto e(x)$.)

<u>Prop</u> (Major arcs) $s \geq 2k+1$. Then

$$\int_{\mathfrak{M}} \widehat{1_X}(\theta)^s e(N\theta)\, d\theta = \mathfrak{S}_{k,s}(N) \cdot N^{\frac{s}{k}-1} + o(N^{\frac{s}{k}-1}).$$

<u>Prop</u> (Minor arcs) $s \geq 100^k$. Then

$$\int_{\mathfrak{m}} \widehat{1_X}(\theta)^s e(N\theta)\, d\theta = o(N^{\frac{s}{k}-1})$$

<u>Summary</u> Prop'ns + bounds of $\mathfrak{S}_{k,s}$ $\Rightarrow$ sol'n of Waring's problem.