Lecture 4: Waring's Problem: the minor arcs

§1 Circle method (continued)

Recall we need to prove 3 things to solve the problem:

Prop 1 (Major arcs) Let $s \geq 2k+1$. $X = \{n^k : n \leq N^{\frac{1}{k}}\}$. Then

$$\int_{\mathfrak{M}} \widehat{1_X}(\theta)^s e(N\theta)\, d\theta = \mathfrak{S}_{k,s}(N) \cdot N^{\frac{s}{k}-1} + o(N^{\frac{s}{k}-1}).$$

Prop 2 (Minor arcs) Let $s \geq 100^k$. Then

$$\int_{\mathfrak{m}} \widehat{1_X}(\theta)^s e(N\theta)\, d\theta = o(N^{\frac{s}{k}-1}).$$

Prop 3 (Singular series) Let $s \geq k^4$. Then

$$1 \ll \mathfrak{S}_{k,s}(N) \ll 1 \quad (\text{i.e. } \mathfrak{S}_{k,s}(N) \asymp 1).$$

§2 Minor arcs

By the triangle inequality,

$$\left| \int_{\mathfrak{m}} \widehat{1_X}(\theta)^s e(N\theta)\, d\theta \right| \leq \sup_{\theta \in \mathfrak{m}} |\widehat{1_X}(\theta)|^s$$

so the Prop 2 will follow from:

Prop 2' (Pointwise estimate) Let $\varepsilon = 100^{-k}$. Then

$$\sup_{\theta \in \mathfrak{m}} |\widehat{1_X}(\theta)| \ll N^{\frac{1}{k}-\varepsilon}.$$

(We will deduce this from a slightly more general bound for exponential sums $\sum_{x \in I} e(P(x))$ (Weyl sums)).

§3 Weyl Sums

Thm (estimate for Weyl sums)

Set $C_k := 10^k$. Let δ be sufficiently small in terms of k.

Suppose that $L > \delta^{-C_k}$. Let $I \subseteq \mathbb{Z}$ be an interval of length $\leq L$.

Let $P:\mathbb{Z}\to\mathbb{R}$, $P(x)=\alpha x^k+\cdots$ poly of deg k.

Suppose $\left|\sum_{x\in I} e(P(x))\right| \geq \delta L$.

Then $\exists\, q \leq \delta^{-C_k}$ s.t. $\|q\alpha\| \leq \delta^{-C_k} L^{-k}$.

dist to the nearest int.

Deduction of Prop 2': Take $I=\{n\leq N^{\frac{1}{k}}\}$, $L=\lfloor N^{\frac{1}{k}}\rfloor$, $\delta=N^{-\varepsilon}$ $(\varepsilon=100^{-k})$.

Then if $\theta\in\mathbb{R}$ satisfies $|\hat{1}_X(\theta)| > \delta N^{\frac{1}{k}}$,

then $\exists\, q\leq \delta^{-C_k} \leq N^{\eta}$ $(\eta=\frac{1}{10k})$ s.t. $\|q\theta\| \leq \delta^{-C_k}L^{-k} \ll N^{\eta-1}$, so $\theta\in\mathfrak{M}$.

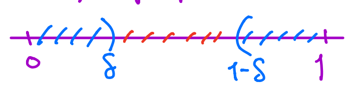
§4 Vinogradov's lemma

The proof of Thm (Weyl sums) makes use of a lemma

on the distribution of $n\alpha$ mod 1.

Philosophy Expect that: α "highly irrational" $\Rightarrow$ (almost) uniform distribution

i.e. $\|\alpha n\|\leq\delta$ for proportion 2δ for integers $n\leq N$.



The next: $\|\alpha n\|$ far from uniformly distributed $\Rightarrow$ α "highly rational".

Lemma (Vinogradov)

$\exists$ absolute const C s.t.:

(*) Suppose (1) $\alpha\in\mathbb{R}$ & $I\subseteq\mathbb{Z}$ with $|I|=N$.

(2) $\delta_1,\delta_2>0$ s.t. $\delta_2>C\delta_1$,

(3) $\exists$ at least $\delta_2 N$ elements $n\in I$, $\|\alpha n\|\leq\delta_1$,

(4) $N\geq \frac{C}{\delta_2}$.

Then $\exists\, 1\leq q\leq \frac{C}{\delta_2}$ s.t. $\|\alpha q\| \leq \frac{C\delta_1}{\delta_2 N}$.

Roughly If $\|\alpha n\| \approx \delta$ for $> 1000\delta N$ integers and $N \gg_\delta 1$,

then $\exists\, q \ll_\delta 1$ s.t. $\|q\alpha\| \ll_\delta \frac{1}{N}$.

For the proof, we start with a well-known lemma:

Theorem (Dirichlet) $\alpha \in \mathbb{R}$, $Q \geq 1$. Then $\exists\, 1 \leq q \leq Q$ s.t. $\|q\alpha\| \leq \frac{1}{Q}$.

Pf: Apply the pigeonhole principle to $\alpha, 2\alpha, \cdots, Q\alpha \pmod 1$.

Proof The proof of Vinogradov's lemma is in steps.

Let $S = \{ n \in I : \|\alpha n\| \leq \delta_1 \}$.

Step 1: Reduction to the case $I = [1, N] \cap \mathbb{Z}$.

(This is just a change of variables).

Step 2: Applying Dirichlet's theorem.

Apply with $Q = 4N$. Thus, $\exists\, 1 \leq q \leq 4N$ s.t. $\|\alpha q\| \leq \frac{1}{4N}$.

Hence $\exists\, a$, coprime to q s.t. $|\alpha - \frac{a}{q}| \leq \frac{1}{4qN}$.

This gives $\|\alpha N\| \leq \|\frac{aN}{q}\| + \frac{1}{4q}$, $n \in S$.

Step 3: Reducing q.
$$\begin{aligned}
&\#\{\text{sol'ns } n \text{ to } \|\tfrac{an}{q}\| \leq \delta_1 + \tfrac{1}{4q}\} \\
&\leq \left(\frac{N}{q} + 1\right) \cdot \#\{1 \leq n \leq q : \|\tfrac{an}{q}\| \leq \delta_1 + \tfrac{1}{4q}\} \\
&\leq \left(\frac{N}{q} + 1\right) \cdot \left(2q\left(\delta_1 + \frac{1}{4q}\right) + 1\right).
\end{aligned}$$

This should be $\geq \delta_2 N$, so with a bit of algebra $q \leq \frac{16}{\delta_2}$.

Step 4 Reducing $\|\alpha\|$.

By Step 3, we have $q \leq \frac{16}{\delta_2}$. So $\delta_1 < \frac{1}{2q}$.

Recalling $|\alpha - \frac{a}{q}| \leq \frac{1}{4qN}$, this gives
$$\left\|\frac{an}{q}\right\| < \frac{1}{q}, \quad n \in S.$$

Thus $S \subseteq q\mathbb{Z} \cap [1, N]$.

Step 5 Finishing the proof.

Let $\theta = \alpha - \frac{a}{q}$. Since $S \subseteq q\mathbb{Z}$, we have $\|\theta n\| = \|\alpha n\|$ for $n \in S$.
But $|\theta| \leq \frac{1}{4Nq}$, so $\|\theta n\| = |\theta n|$ for all $n \leq N$. Thus

$$|\theta n| \leq \delta_1, \quad n \in S.$$

But since $|S| \geq \delta_2 N$ and $S \subseteq q\mathbb{Z}$,

$\rightsquigarrow \quad \exists\, n_0 \in S$ s.t. $|n_0| \geq \delta_2 q N$.

Choosing $n = n_0$ here,

we get $|\theta| \leq \frac{\delta_1}{q \delta_2 N} \Rightarrow \|\alpha q\| \leq \|\theta q\| \leq \frac{\delta_1}{\delta_2 N}$. □

§5 Proof of Weyl sum estimate

We need one more ingredient.

Lemma Let X be finite and $b : X \to \mathbb{C}$ s.t. $|b(x)| \leq 1$ ($\forall x \in X$).

Suppose $\left|\sum_{x \in X} b(x)\right| \geq \varepsilon |X|$.

Then $\exists \geq \frac{\varepsilon}{2}|X|$ values of $x \in X$ for which $|b(x)| \geq \frac{\varepsilon}{2}|X|$.

Pf. Argue by contradiction. ☺

Prove by induction. Start with $k=1$.

~~Proof for $k=1$~~ $P(x) = \alpha x + \beta$ linear.

By the geom sum formula,

$$\left|\sum_{x \in I} e(P(x))\right| = \left|\sum_{j=0}^{|I|-1} e(\alpha j)\right| = \left|\frac{1 - e(\alpha |I|)}{1 - e(\alpha)}\right| \leq \frac{2}{|1 - e(\alpha)|} \ll \frac{1}{\|\alpha\|}.$$

Hence if LHS $> \delta L$, then $\|\alpha\| \ll \delta^{-1} L^{-1}$.

<u>Assuming thm for $k-1$</u>

<u>Step 1</u> Square out and look at discrete derivatives.

By assumption

$$\left|\sum_{x \in I} e(P(x))\right|^2 \geq \delta^2 L^2 \Rightarrow \left|\sum_{x,y \in I} e(P(x) - P(y))\right| \geq \delta^2 L^2.$$

Take $h = y - x \rightsquigarrow \partial_h f(x) = f(x+h) - f(x)$,

$$\Big|\sum_{|h|\le L,\, x\in I_h} e(\partial_h P(x))\Big| \ge \delta^2 L^2, \quad I_h = I \cap (I-h).$$

By the averaging lemma, this gives

$$\exists \ge \delta^2 L/6 \text{ values of } |h| \le L \text{ s.t. } \Big|\sum_{x\in I_h} e(\partial_h P(x))\Big| \ge \frac{\delta^2 L}{6}.$$

Since $L > 100\delta^{-2}$, the contribution of $h=0$ is small,

so there are $\delta^2 L/18$ positive (or negative) h with this property.

Step 2 Applying the induction assumption.

Let $H = \{h : \text{ of size} \ge \delta^2 L/18\}$.

Note that (crucially)

$$\partial_h P(x) = k\alpha x^{k-1} + \cdots \quad \text{poly of deg } k-1.$$

$$\Rightarrow \forall h \in H \ \exists q_h \ll \delta^{-2C_{k-1}} \text{ s.t. } \|k h q_h \alpha\| \ll \delta^{-2C_{k-1}} \cdot L^{-(k-1)}.$$

By pigeonholing, $\exists H' \subset H$ of size $\gg \delta^{2+2C_{k-1}} L$

s.t. $q_h := q'$ is const for $h \in H'$.

Step 3 Applying Vinogradov's lemma.

Apply with $\alpha' = kq'\alpha$, $\delta_1 = C_1 \delta^{-2C_{k-1}} L^{-(k-1)}$, $\delta_2 = C_2 \delta^{2+2C_{k-1}}$

(we have $\delta_2 > C\delta_1$, since $C_k > 2 + 4C_{k-1}$).

$$\Rightarrow \exists q'' \ll \delta_2^{-1} \ll \delta^{-2-2C_{k-1}} \text{ s.t. } \|\alpha q''\| \ll \frac{\delta_1}{\delta_2 L} \ll \delta^{-2-4C_{k-1}} L^{-k}.$$

Letting $q = kq'q''$ and recalling $C_k > 2 + 4C_{k-1}$. Done $\square$