Some consequences of mod p multiplicity one for Shimura curves

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<u>Outline</u> (1) local-global compatibility for complete coh.

(2) Categorical mod p Langlands for non split inner-forms of $GL_2(\mathbb{Q}_p)$.

<u>Notations</u> $F/\mathbb{Q}$ tot real, $\deg = f$.

p inert in F.

Coeff: $E/\mathbb{Q}_p$ finite, $E \supseteq \mathcal{O} \twoheadrightarrow k$.

∞_v: infin place of F.

D/F quaternion, non-split at all infin places except possibly ∞_v.

$\Rightarrow$ Either $D_p \simeq M_2(\mathbb{Q}_p)$ (split) or division algebra (non-split).

- When D_∞ is split (indefinite)

X_n = Shimura curve of tame level K^p and

$K_p = K_n = \ker(GL_2(\mathcal{O}_{F_p}) \to GL_2(\mathcal{O}_{F_p}/p^n))$

or $1 + \mathfrak{m}_{D_p}^n$.

Define $\pi_n = H^1_{et}(X_n \times_F \bar{F}, k)$ and $\pi = \varinjlim_n \pi_n$.

- When D_∞ is non-split (definite)

Let π_n = space of mod p modular form $D^\times(\mathbb{A}_F^\infty)$

of level $K^p K_n$ and $\Pi = \varinjlim_n \Pi_n$.

<u>Then</u> Π has commuting action of $D_p^\times$ & Hecke alg
$\mathbb{T} = k[T_v, S_v^{\pm 1} \mid v \text{ s.t. } K_v^p \text{ hyperspecial}]$.
and Gal_F in the indef case.

<u>Expectation</u> Let m, m' be max'l ideals of $\mathbb{T}$
s.t. $\Pi[m], \Pi[m']$ are nonzero.
Assume that $r_m, r_{m'} : \mathrm{Gal}_F \longrightarrow GL_2(k)$ satisfy
$r_m|_{\mathrm{Gal}_{F_p}} \simeq r_{m'}|_{\mathrm{Gal}_{F_p}}$
(and K^p is minimal for m and m'.)
Then $\Pi[m] \cong \Pi[m']$ as $D_p^\times$-rep'n.

<u>Partial results</u> Assume D_p is split,
m, m' non-Eisenstein s.t. $r_m, r_{m'}$ satisfy
Taylor-Wiles assumption.
If $r_m|_{\mathrm{Gal}_{F_p}} \simeq r_{m'}|_{\mathrm{Gal}_{F_p}}$ sufficiently generic, then
(1) $\mathrm{Soc}_{GL_2(\mathcal{O}_{F_p})} \Pi_m \simeq \mathrm{Soc}_{GL_2(\mathcal{O}_{F_p})} \Pi_{m'}$
where $\Pi_m = \Pi[m]$ or $\mathrm{Hom}_{\mathrm{Gal}_F}(r_m, \Pi[m])$
(def'te) (indef'te).
(2) $\Pi_m^{K_1} \simeq \Pi_{m'}^{K_1}$ as $GL_2(\mathbb{F}_{p^f})$-rep's.
(3) $(\Pi_m^{I_{w_1}} \longrightarrow \Pi_m^{K_1}) \cong (\Pi_{m'}^{I_{w_1}} \longrightarrow \Pi_{m'}^{K_1})$
(4) $\dim_{GL_2(F_p)} \Pi_m = \dim_{GL_2(F_p)} \Pi_{m'} = f$.

One of the main input is mod p multi one:

If σ is an irred $k[GL_2(\mathbb{F}_{p^f})]$-rep (Serre wt)

then $\dim \mathrm{Hom}_{GL_2(\mathcal{O}_{F_p})}(\sigma, \Pi_m) = 1$ or 0.

More generally, this holds after replacing σ by $\mathrm{Proj}_{k[GL_2(\mathbb{F}_{p^f})]}(\sigma)$.

<u>Thm</u> Assume $D_p^\times$ non-split.

Let $\psi: \mathcal{O}_{D_p}^\times \longrightarrow k^\times$ regular char ($\psi \neq \psi^{p^f}$).

Then $\dim \mathrm{Hom}_{\mathcal{O}_{D_p}^\times}(\psi, \Pi_m) = 1$ or 0.

(*) Work in progress (with Le Hung):

Use this to show that $\dim_{D_p^\times} \Pi_m = f$.

Multi one in split case:

Construct a "patched module".

$$M_\infty \circlearrowleft \mathcal{O}[GL_2(F_p)] \times R_{\bar\rho}^\square, \quad \bar\rho = \eta_m|_{\mathrm{Gal}_{F_p}}$$

such that

(a) For every lattice τ° in an $E(GL_2(\mathbb{F}_{p^f}))$-irrep,

we have that $\underset{\substack{\| \\ M_\infty(\tau^\circ)}}{\mathrm{Hom}_{GL_2(\mathcal{O}_{F_p})}(\tau^\circ, M_\infty^\vee)^\vee}$ is supported on $R_{\bar\rho}^\tau$,

where $R_{\bar\rho}^\tau = \begin{pmatrix} \text{pos semistable def'te ring} \\ \text{w/ minimal HT type and inertia type } \tau \end{pmatrix}$.

(b) $M_\infty(\tau^\circ)/m_{\bar\rho} = \mathrm{Hom}_{GL_2(\mathcal{O}_{F_p})}(\tau^\circ \otimes_{\mathcal{O}} k, \Pi_m)^\vee$

then show that $M_\infty(\tau^\circ) \cong R_{\bar\rho}^\tau$ whenever τ°

has irred cosocle (induction on Serre wt).

Two ways to fix this problem

(1) p-adic uniformization

Let B/F be obtained from D by switch inv at p and ∞_v.

For all regular ψ,

$\exists$ lattice $\tau^\circ(\psi) \subseteq$ DL-rep attached to ψ

s.t. $\dim \mathrm{Hom}_{\mathcal{O}_{\mathbb{F}_p}^\times}(\psi, \Pi_p[m])$

$\leq \dim \mathrm{Hom}_{GL_2(\mathcal{O}_{F_p})}(\tau^\circ(\psi) \otimes \tau(\psi)^{pf}, \Pi_B[m])$

(2) Construction using the BK mod of

the p-torsion in the conn Néron model.

§ Applications to categorical mod p Langlands

Conj (Emerton-Gee-Hellmann)

$\exists$ fully faithful, exact functor

$\mathcal{A} : D^b_{fp}(GL_2(\mathbb{Q}_{p^f}), k) \longrightarrow D^b_{coh}(\mathcal{X}_{2,\mathbb{Q}_{p^f}})$

s.t. (1) if τ° lattice in $E[GL_2(\mathbb{F}_p)]$ - irrep then

$\mathcal{A}(c\mathrm{Ind}_{KZ}^{GL_2(\mathbb{Q}_{p^f})}(\tau^\circ \otimes k))$ is concentrated in deg 0,

and supp on $\mathcal{X}^\tau_{2,\mathbb{Q}_{p^f}} \times_{\mathcal{O}} k$.

(2) $\mathcal{A}$ compatible with $M_\infty(\tau)$ after pulling back

to versal rings.

By (2), multi one $\Rightarrow$ the sheaf in (1)

are invertible over their support.

Conj in nonsplit case leads (1).

$\forall$ generic $\chi : \mathcal{O}_{D_p}^{\times} \to k^{\times}$, $\exists$ invertible sheaf $L(\chi)$

on $\mathcal{X}_{2,\mathbb{Q}_p^f}^{\tau(\chi)} \times_{\mathcal{O}} k$ s.t.

$$\mathrm{Ext}^{\bullet}_{D_p^{\times}}(c\text{-}\mathrm{Ind}(\oplus \chi), c\text{-}\mathrm{Ind}(\oplus \chi')) \cong \mathrm{Ext}^{\bullet}_{\mathcal{X}_{2,\mathbb{Q}_p^f}}(\oplus L(\chi), \oplus L(\chi')).$$