

Meromorphic vector bundles on FF curve

Ian Gleason

G/\mathbb{Q}_p reductive, \mathcal{G}/\mathbb{Z}_p parahoric.

Analytically $\text{Bun}_G = \{G\text{-bundles} / X_{\text{FF}}\}$

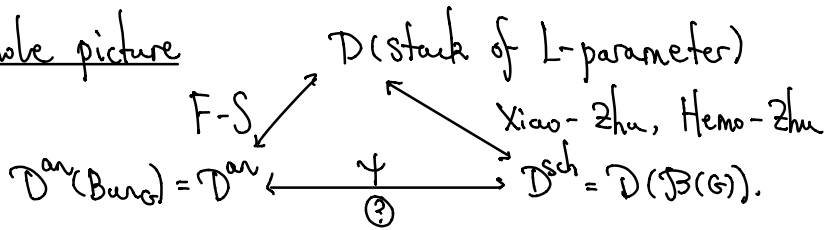
$$\text{Sht}_G^{\text{an}} = \left\{ \begin{array}{l} \mathcal{G}\text{-bundles} / \mathcal{Y}_{(0, \infty)} \\ \mathbb{F}: \varphi^* \mathcal{E} \dashrightarrow \mathcal{E} \\ \text{with pole at } p=0 \end{array} \right\}$$

Schematically $\mathcal{B}(G) = LG //^{\varphi} LG$,

$$\text{Sht}_G^{\text{sch}} = LG //^{\varphi} L^+ \mathcal{G}, \quad L^+ \mathcal{G}(\text{Spec } R)$$

where $LG(R) = G(W(R)[\frac{1}{p}])$, $L^+ \mathcal{G}(R) = \mathcal{G}(W(R))$.

Whole picture



Goal/Dream (a) Construct ψ directly.

(b) Explicit about it.

Observation $\text{Bun}_G(C) = \mathcal{B}(G)$, $\mathcal{B}(G)(\text{Spec } C) = \mathcal{B}(G)$.

by Fargues

$\forall b \in G$, get $i_b: \mathcal{B}(G)_b \rightarrow \mathcal{B}(G)$, $j_b: \text{Bun}_G^b \rightarrow \text{Bun}_G$.

Known $D(\text{Rep } J_b) \simeq D^{\text{an}}(\text{Bun}_G^b) \simeq D^{\text{sch}}(\mathcal{B}(G)_b) \simeq D^{\text{an}}(I^*/J_b I)$.

$\hookrightarrow D_{b,!}^{\text{sch}} = i_{b,!} D(\text{Rep } J_b) \subseteq D^{\text{sch}}$, $D_{b,!}^{\text{an}} = j_{b,!} D(\text{Rep } J_b) \subseteq D^{\text{an}}$.

Naïve guess $\psi(D_{b,!}^{sch}) = D_{b,!}^{an}$ (wrong).

b/c $|\beta(G)| = \beta(G)$ [Rapoport-Richartz, HeJ]

$|\text{Bun}_G| = \beta(G)^p$ [Hansen, Hamann, Viehmann].

2nd guess $\psi(D_{b,!}^{sch}) = D_{b,*}^{an}$ (wrong)

$\psi(D_{b,*}^{sch}) = D_{b,!}^{an}$ (unknown)

3rd guess $D_V^{sch}(D_{b,*}^{sch}) = D_{b,!}^{sch}$

$\psi(D_{b,!}^{sch}) = D_{BZ}^{an}(D_{b,!}^{an}) = D_{M_b}^{an}$

\rightsquigarrow



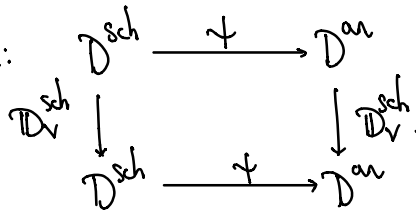
Reasonably $D_{M_b}^{an} = \sigma_b! \gamma_b^* D(\text{Rep } J_b)$.

Conj (a) ψ exists

(b) $\psi(D_{b,!}^{sch}) = D_{b,*}^{an}$

(c) $\psi(D_{b,*}^{sch}) = D_{b,!}^{an}$

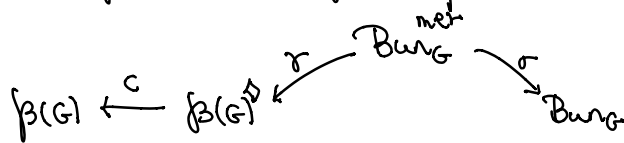
(d) Comm diagram:



Approach Scholze analytification

$$c^*: D^{sch}(\beta(G)) \longrightarrow D^{an}(\beta(G)^\diamond)$$

note: $\beta(G)^\diamond(R, R^+) = \beta(G)(\text{Spec } R)$.



$$\rightsquigarrow \psi = \sigma! \gamma^* c^*$$

Thm A (Gleason-Ivanov)

$$(1) \quad \begin{array}{ccc} & \cup \sigma_b & \\ & \curvearrowright & \\ \bigcup_{b \in B(G)} \mathcal{M}_b & \xrightarrow{\quad} & \text{Bun}_G^{\text{mer}} \xrightarrow{\sigma} \text{Bun}_G \\ \downarrow \cup \gamma_b \quad \Gamma & & \downarrow \gamma \\ \bigcup_{b \in B(G)} \beta(G)_b^\diamond & \xrightarrow{\cup \iota_b^\diamond} & \beta(G)^\diamond \end{array}$$

(2) (Strata local-global compatibility)
 $\sigma_! \delta^* \iota_{b,!}^\diamond = \sigma_b! \delta_b^*$

Thm B For torsion coeff, r^*c^* is fully faithful.

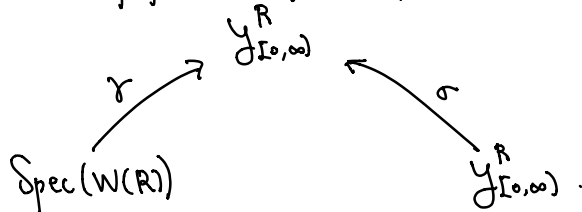
Prop $S = \text{Spa}(R, R^\dagger)$. The following cats are equiv:

- (a) IsoShtukas over S with pole at $p=0$.
- (b) φ -mods over bounded Robba ring R_S^{bd} } $\text{Sht}[\frac{1}{p}]$.

E.g. $M = \begin{pmatrix} \frac{1}{p} & \frac{1}{[p]} \\ 0 & \frac{1}{p} \end{pmatrix} \in M_{2 \times 2}(W(\mathbb{Q}_p)[\frac{1}{p}]).$

$\Rightarrow M$ is σ -conj to $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ in $H^0(Y_{(0,\infty)}, \mathbb{Q})$
 but NOT in $H^0(Y_{[0,\infty)}, \mathbb{Q})[\frac{1}{p}]$.

Fact $\text{Bun}_G^{\text{mer}} = (\text{sheafification of } \text{Sht}[\frac{1}{p}])$



Def A semistable filtered v.b. over X_{FF}^R

is an increasing fil'n $\{E_{\leq \lambda} \mid \lambda \in \mathbb{Q}\}$ of v.b.s on X_{FF}^R
 s.t. $E_\lambda := E_{\leq \lambda} / E_{< \lambda}$ is semistable of slope λ .

Denote $\text{Fil}^{\text{ss}}(R, R^+) = \text{cat of s.s. fil'd v.b.}$

Def $(\text{Bun}_G^{\text{mer}})^{\text{loc}} \subset \text{Bun}_G^{\text{mer}}$
 substack with loc const geometric Newton polygon.

Prop $S = \text{Spa}(R, R^+)$ is a product of pts, then

(a) $\text{Sht}[\frac{1}{p}](S) = \text{Bun}_G^{\text{mer}}(S)$

(b) $(\text{Bun}_G^{\text{mer}})^{\text{loc}}(S) = \text{Fil}^{\text{ss}}(S)$.

General question If \mathcal{F} is a small v -sheaf,
 what is $\text{Bun}_G(\mathcal{F})$?

Let A perfect ring / \mathbb{F}_p , $\pi \in A$ non-zero divisor.

$A = \hat{A}_\pi$ algebraically.

$B = \hat{A}_\pi$ π -adic rep, $R = B[\frac{1}{\pi}]$, $R^+ = B$.

Thm (Anschütz, Pappas-Rapoport, Gleason-Ivanov, Grütthe).

(a) $\text{Bun}_G(\text{Spd}(A, A)) = \mathcal{B}(G)(\text{Spec } A)$

(b) $\text{Sht}_G^{\text{an}}(\text{Spd}(A, A)) = \text{Sht}_G^{\text{sch}}(\text{Spec } A)$

(c) $\text{Bun}_G(\text{Spd}(B, B)) = G\text{-}\varphi\text{-mod} / \mathcal{Y}_{(0, \infty)}^R$

(d) $\text{Bun}_G(\text{Spd}(A[\frac{1}{\pi}], A)) = G\text{-IsoShtukas} / \mathbb{R}$.

Thm (He, Viehmann) $|\mathcal{B}(G)| = |\text{Bun}_G|^{\text{pp}}$.

Sketch V rk 1 valuation ring

Cutlines: \textcircled{g} = generic pt. \textcircled{s} = special pt.

$$(1) \text{Spec } V = \begin{array}{ccc} \textcircled{g} & & \\ \downarrow & & \downarrow \\ \textcircled{s} & & \\ \beta(G) & \xrightarrow{bg} & bs \\ \downarrow & & \\ bg, bs & & \end{array}$$

$$(2) \text{Spa}(V, V) = \begin{array}{ccc} \textcircled{g} & & \\ \downarrow & & \\ \textcircled{h} & \longrightarrow & \textcircled{s} \text{ horizontal} \end{array}$$

$$(3) \text{Spd}(V, V) \xrightarrow{(\text{mer})} \text{Bun}_G$$

$$\begin{array}{ccc} \textcircled{g} & & \textcircled{a} \\ \downarrow & \swarrow \text{mer} & \searrow \text{formal} \\ \textcircled{m} & & \textcircled{s} \\ \underbrace{\hspace{10em}} & & \\ \text{Spd}(A[\frac{1}{\omega}], A) & \text{Spd}(R^+, R^+) & \end{array} \quad bs \geq bg.$$

Thm C $\text{Bun}_G^{\text{mer}} = \beta(G)^{\dagger}$
 where $\beta(G)^{\dagger}(R, R^+) = \beta(G)(\text{Spec } R)$.