Meromorphic vector bundles on FF curve

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$G/\mathbb{Q}_p$ reductive, $\mathcal{G}/\mathbb{Z}_p$ parahoric.

<u>Analytically</u> $\mathrm{Bun}_G = \{G\text{-bundles}/X_{FF}\}$

$$\mathrm{Sht}^{an}_{\mathcal{G}} = \left\{ \begin{array}{l} \mathcal{G}\text{-bundles}/\mathcal{Y}_{[0,\infty)} \\ \Phi : \varphi^*\mathcal{E} \dashrightarrow \mathcal{E} \\ \text{with pole at } p=0 \end{array} \right\}$$

<u>Schematically</u> $\mathfrak{B}(G) = LG /\!/^{\varphi} LG$,

$$\mathrm{Sht}^{sch}_{\mathcal{G}} = LG /\!/^{\varphi} L^+\mathcal{G}. \qquad L^+\mathcal{G}(\mathrm{Spec}\, R)$$

where $LG(R) = G(W(R)[\frac{1}{p}])$, $L^+\mathcal{G}(R) = \mathcal{G}(W(R))$,

<u>Whole picture</u>

$$\begin{tikzcd} & D(\text{stack of L-parameter}) \arrow[dl, leftrightarrow, "F\text{-}S"'] \arrow[dr, leftrightarrow, "\text{Xiao-Zhu, Hemo-Zhu}"] & \\ \mathcal{D}^{an}(\mathrm{Bun}_G) = \mathcal{D}^{an} \arrow[rr, leftrightarrow, "\Psi", "?"'] & & \mathcal{D}^{sch} = D(\mathfrak{B}(G)). \end{tikzcd}$$

<u>Goal / Dream</u> (a) Construct Ψ directly.

(b) Explicit about it.

<u>Observation</u> $\underbrace{\mathrm{Bun}_G(C) = B(G)}_{\text{by Fargues}}$, $\mathfrak{B}(G)(\mathrm{Spa}\, C) = B(G)$.

$\forall b \in G$, get $i_b : \mathfrak{B}(G)_b \longrightarrow \mathfrak{B}(G)$, $j_b : \mathrm{Bun}_G^b \longrightarrow \mathrm{Bun}_G$.

<u>Known</u> $D(\mathrm{Rep}\, J_b) \simeq \mathcal{D}^{an}(\mathrm{Bun}_G^b) \simeq \mathcal{D}^{sch}(\mathfrak{B}(G)_b) \simeq \mathcal{D}^{an}([*/J_b])$.

$\rightsquigarrow$ $\mathcal{D}^{sch}_{b,!} = i_{b!} D(\mathrm{Rep}\, J_b) \subseteq \mathcal{D}^{sch}$, $\mathcal{D}^{an}_{b,!} = j_{b,!} D(\mathrm{Rep}\, J_b) \subseteq \mathcal{D}^{an}$.

<u>Naïve guess</u> $\Psi(D^{sch}_{b,!}) = D^{an}_{b,!}$ (wrong).

b/c $|\mathfrak{B}(G)| = B(G)$ [Rapoport–Richartz, He]

$|\mathrm{Bun}_G| = B(G)^{op}$ [Hansen, Hamann, Viehmann].

<u>2nd guess</u> $\Psi(D^{sch}_{b,!}) = D^{an}_{b,*}$ (wrong)

$\Psi(D^{sch}_{b,*}) = D^{an}_{b!}$ (unknown)

<u>3rd guess</u> $\mathbb{D}^{sch}_V(D^{sch}_{b,*}) = D^{sch}_{b,!}$

$\Psi(D^{sch}_{b,!}) = \mathbb{D}_{BZ}(D^{an}_{b,!}) = D^{an}_{\mathcal{M}_b}$

$\rightsquigarrow$
$$\begin{tikzcd} & \mathcal{M}_b \arrow[dl, "\gamma_b"'] \arrow[dr, "\sigma_b"] & \\ {[*/J_b]} & & \mathrm{Bun}_G \end{tikzcd}$$

<u>Reasonably</u> $D^{an}_{\mathcal{M}_b} = \sigma_{b!}\gamma_b^* D(\mathrm{Rep}\, J_b)$.

<u>Conj</u> (a) Ψ exists

(b) $\Psi(D^{sch}_{b,!}) = D^{an}_{b,*}$

(c) $\Psi(D^{sch}_{b,*}) = D^{an}_{b,!}$

(d) Comm diagram:
$$\begin{tikzcd} D^{sch} \arrow[r, "\Psi"] \arrow[d, "\mathbb{D}^{sch}_V"'] & D^{an} \arrow[d, "\mathbb{D}^{sch}_V"] \\ D^{sch} \arrow[r, "\Psi"] & D^{an} \end{tikzcd}$$

<u>Approach</u> Scholze analytification

$c^*: D^{sch}(\mathfrak{B}(G)) \longrightarrow D^{an}(\mathfrak{B}(G)^\diamond)$

<u>note</u>: $\mathfrak{B}(G)^\diamond(R, R^+) = \mathfrak{B}(G)(\mathrm{Spec}\, R)$.

$$\begin{tikzcd} & & \mathrm{Bun}_G^{mer} \arrow[dl, "\gamma"'] \arrow[dr, "\sigma"] & \\ \mathfrak{B}(G) & \mathfrak{B}(G)^\diamond \arrow[l, "c"'] & & \mathrm{Bun}_G \end{tikzcd}$$

$\rightsquigarrow \Psi = \sigma_! \gamma^* c^*$

Thm A (Gleason–Ivanov)

(1)

$$\begin{tikzcd}
\bigsqcup_{b\in B(G)} \mathcal{M}_b \arrow[r] \arrow[d, "\sqcup \gamma_b"'] \arrow[rr, bend left, "\sqcup \sigma_b"] \arrow[dr, phantom, "\ulcorner", very near start] & \mathrm{Bun}_G^{\mathrm{mer}} \arrow[r, "\sigma"] \arrow[d, "\gamma"] & \mathrm{Bun}_G \\
\bigsqcup_{b\in B(G)} \beta(G)_b^{\diamondsuit} \arrow[r, "\sqcup i_b^{\diamondsuit}"] & \beta(G)^{\diamondsuit} &
\end{tikzcd}$$

(2) (Strata local-global compactibility)

$$\sigma_! \gamma^* i^{\diamondsuit}_{b,!} = \sigma_{b!} \gamma_b^* .$$

Thm.B For torsion coeff, $\gamma^* c^*$ is fully faithful.

Prop $S = \mathrm{Spa}(R, R^+)$. The following cats are equiv:

(a) IsoShtukas over S with pole at $p=0$.

(b) φ-mods over bounded Robba ring $\mathcal{R}_S^{bd}$

$\Big\}\ \mathrm{Sht}[\frac{1}{p}]$.

E.g. $M = \begin{pmatrix} \frac{1}{p} & \frac{1}{[\varpi]} \\ 0 & \frac{1}{p} \end{pmatrix} \in M_{2\times 2}(W(\mathcal{O}_C)[\frac{1}{p[\varpi]}])$.

$\Rightarrow$ M is σ-conj to $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ in $H^0(\mathcal{Y}_{(0,\infty)}, \mathcal{O})$

but NOT in $H^0(\mathcal{Y}_{[0,\infty)}, \mathcal{O})[\frac{1}{p}]$.

Fact $\mathrm{Bun}_G^{\mathrm{mer}} = (\text{sheafification of } \mathrm{Sht}[\frac{1}{p}])$

$$\begin{tikzcd}
 & \mathcal{Y}^R_{[0,\infty)} & \\
\mathrm{Spec}(W(R)) \arrow[ur, "\gamma"] & & \mathcal{Y}^R_{[0,\infty)} . \arrow[ul, "\sigma"']
\end{tikzcd}$$

Def A <u>semistable filtered v.b. over X_{FF}^R</u> is an increasing fil'n $\{\mathcal{E}_{\le\lambda}\}_{\lambda\in\mathbb{Q}}$ of v.b.s on X_{FF}^R s.t. $\mathcal{E}_\lambda := \mathcal{E}_{\le\lambda}/\mathcal{E}_{<\lambda}$ is semistable of slope λ.

Denote $\mathrm{Fil}^{ss}(R,R^+)$ = cat of s.s. fil'd v.b.

<u>Def</u> $(\mathrm{Bun}_G^{mer})^{loc} \subset \mathrm{Bun}_G^{mer}$
substack with loc const geometric Newton polygon.

<u>Prop</u> $S = \mathrm{Spa}(R,R^+)$ is a product of pts. then
(a) $\mathrm{Sht}[\frac{1}{p}](S) = \mathrm{Bun}_G^{mer}(S)$
(b) $(\mathrm{Bun}_G^{mer})^{loc}(S) = \mathrm{Fil}^{ss}(S)$.

<u>General question</u> If $\mathcal{F}$ is a small v-sheaf,
what is $\mathrm{Bun}_G(\mathcal{F})$?

Let A perfect ring / $\mathbb{F}_p$, $\pi \in A$ non-zero divisor.
$A = \tilde{A}_\pi$ algebraically.
$B = \hat{A}_\pi$ π-adic rep, $R = B[\frac{1}{\pi}]$, $R^+ = B$.

<u>Thm</u> (Anschütz, Pappas-Rapoport, Gleason-Ivanov, Grüthge).
(a) $\mathrm{Bun}_G(\mathrm{Spd}(A,A)) = \mathfrak{B}(G)(\mathrm{Spec}\,A)$
(b) $\mathrm{Sht}_{\mathcal{G}}^{an}(\mathrm{Spd}(A,A)) = \mathrm{Sht}_{\mathcal{G}}^{sch}(\mathrm{Spec}\,A)$
(c) $\mathrm{Bun}_G(\mathrm{Spd}(B,B)) = G-\varphi\text{-mod} / \mathcal{Y}^R_{(0,\infty]}$
(d) $\mathrm{Bun}_G(\mathrm{Spd}(A[\frac{1}{\pi}],A)) = G\text{-IsoShtukas} / R$.

<u>Thm</u> (He, Viehmann) $|\mathfrak{B}(G)| = |\mathrm{Bun}_G|^{op}$.
<u>Sketch</u> V rk 1 valuation ring

Outlines: g = generic pt. s = special pt.

(1) $\operatorname{Spec} V = \begin{tikzcd} g \arrow[d] \\ s \end{tikzcd}$

$$\begin{tikzcd} \operatorname{Spec} V \arrow[d] \\ \mathfrak{B}(G) \end{tikzcd} \qquad b_g \longrightarrow b_s$$

$$\mathfrak{B}(G) \ni b_g, b_s$$

(2) $\operatorname{Spa}(V, V) = \begin{tikzcd} g \arrow[d] & \\ h \arrow[r] & s \end{tikzcd}$ horizontal

(3) $\operatorname{Spd}(V, V) \; (\longrightarrow \mathrm{Bun}_G)$

$$\begin{tikzcd} g \arrow[d] & & a \arrow[dll, "\text{mero}"'] \arrow[d, "\text{formal}"] \\ m & & s \end{tikzcd}$$

$$\underbrace{m}_{\operatorname{Spd}(A[\frac{1}{\varpi}], A)} \qquad \underbrace{s}_{\operatorname{Spd}(R^+, R^+)} \qquad b_s \geq b_g.$$

<u>Thm C</u> $\mathrm{Bun}_G^{\mathrm{mer}} = \mathfrak{B}(G)^{\dagger}$

where $\mathfrak{B}(G)^{\dagger}(R, R^+) = \mathfrak{B}(G)(\operatorname{Spec} R^+)$.