What can categorical local Langlands do for YOU?

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Setup Fix $E/\mathbb{Q}_p$, G/E conn red quasi-split.
Fix also a pinning, a nontriv char $\psi : E \to \overline{\mathbb{Q}}_\ell^\times$ ($\ell \neq p$ fixed).

Have Bun_G = moduli stack of G-bundles on the FF curve
$$= \coprod_{b \in B(G)} \mathrm{Bun}_G^b \approx [*/G_b(E)].$$

$$\rightsquigarrow \quad D(\mathrm{Bun}_G) = D_{lis}(\mathrm{Bun}_G, \overline{\mathbb{Q}}_\ell) \circlearrowleft \mathbb{D}_{\mathrm{Verd}}, \mathbb{D}_{BZ}$$

$$\downarrow \vdash \uparrow \qquad i_{b\#} \uparrow \vdash \downarrow i_b^* \vdash \uparrow i_{b*} \qquad \text{Bernstein-Zelevinsky.}$$

$$D_{lis}(\mathrm{Bun}_G^b, \overline{\mathbb{Q}}_\ell) \cong D(G_b(E), \overline{\mathbb{Q}}_\ell) \circlearrowleft \mathbb{D}_{\mathrm{Sm}}, \mathbb{D}_{\mathrm{coh}}$$

Important For $? \in \{!, \#, *\}$,
$$i_{b,?}^{\mathrm{ren}} A = i_{b!}(A \otimes \delta_b^{1/2}[-\langle 2\rho_G, \nu_b \rangle])$$
modulus char of dynamic parabolic of ν_b.

$\mathrm{Par}_G = Z^1(W_E, \hat{G})_{\overline{\mathbb{Q}}_\ell} / \hat{G}$ stack of L-parameters
$$q \downarrow$$
$X_G^{\mathrm{Spec}} = Z^1(W_E, \hat{G})_{\overline{\mathbb{Q}}_\ell} /\!/ \hat{G}$ Coarse moduli of semisimple L-parameters.

$$q^{-1}(x_\phi) = \bigcup_{\phi' \sim \phi} BS_{\phi'}.$$

Fargues-Scholze There's a natural $\mathbb{Q}$-action of $\mathrm{Perf}(\mathrm{Par}_G)$ on $D(\mathrm{Bun}_G)$ extending the action of Hecke operators.

Conj There is a natural equiv

$$\mathrm{Coh}^{qc}(\mathrm{Par}_G) \cong \mathcal{D}(\mathrm{Bun}_G)^{\underset{\text{cpt objs}}{\omega}}$$

whose restriction to $\mathrm{Perf}^{qc}(\mathrm{Par}_G)$ is given by acting on $i_{1!}\underbrace{c\text{-}\mathrm{Ind}_{\mathcal{U}(E)}^{G(E)}\psi}_{\text{Whittaker}}$.

Question What is this have to do with ACTURAL local Langlands?

Most optimistic guess This equiv is t-exact for some t-str on both sides $\Rightarrow$ bij on irred objects.

note
- Irred obj on Bun_G side $\simeq (b,\pi)$ where $b \in B(G)$ & π sm irrep of $G_b(E)$.
- Irred obj on Par_G side $\simeq (\phi, P)$ where ϕ Frob ss parameter & $P \in \mathrm{Irr}(S_\phi)$.

Starting point A bij between such pairs $(b,\pi) \longleftrightarrow (\phi, P)$ should exist canonically and depends only on the choice of Whittaker datum.

Recall $B(G)_{\mathrm{basic}}$, LLC (Kaletha, Kottwitz)

Given ϕ, expect a natural bijection,

$$\begin{array}{ccc} \coprod_{b\in B(G)_{bas}} \Pi_\phi(G_b) & \xrightarrow{\sim} & \mathrm{Irr}(S_\phi^\natural) \\ \downarrow & & \downarrow \\ B(G)_{bas} & \xrightarrow[\sim]{\text{Kottwitz } \kappa} & X^*(Z(\hat{G})^\Gamma) \end{array} \qquad S_\phi^\natural = S_\phi / (\hat{G}_{der} \cap S_\phi)^\circ.$$

depends only on Whittaker datum.

Thm (Bertoloni Meli - Oi)

Assume $B(G)_{bas}$ LLC for G and all of its standard Levi subgp.

Then $\exists$ a natural bij

$$\underbrace{\coprod_{b \in B(G)} \Pi_\phi(G_b)}_{\substack{\uparrow \\ \text{fiber of } \Pi(G_b) \to \Phi(G_b) \to \Phi(G)}} \xrightarrow[\sim]{\iota_\phi} \mathrm{Irr}(S_\phi)$$

Thus, varying ϕ, get a canonical bij

$$\{\text{pairs } (b, \pi)\} \longleftrightarrow \{(\phi, P)\}.$$

Def'n A ss L-parameter ϕ is generous if $q^{-1}(x_\phi) = BS_\phi$

generic + nice to you :)

e.g. $G = GL_2$, ϕ is generous $\iff \phi \simeq \begin{pmatrix} \chi & \\ & \chi \end{pmatrix}$ or $\begin{pmatrix} \chi & \\ & \chi|\cdot| \end{pmatrix}$.

Prop (1) If ϕ is generous, then $\mathring{S}_\phi$ is a torus

(2) If ϕ is generous, then $i_\phi : BS_\phi \hookrightarrow \mathrm{Par}_G$ is a reg immersion & Par_G is sm in a nbhd of it.

Conj 1 Let ϕ be a generous L-parameter, and

$$(b, \pi) \longleftrightarrow (\phi, P)$$

match under BM-O bij. Then

$$i_{\phi *} P * i_{1,!} W_\psi \cong i_{b!}^{\text{ren}} \pi.$$

Conj 2 Suppose $(b, \pi) \longleftrightarrow (\phi, P)$ with ϕ generous.

Then $\quad i_{b!}^{\text{ren}} \pi \simeq i_{b*}^{\text{ren}} \pi \simeq i_{b\natural}^{\text{ren}} \pi.$

Conj 3 Let ϕ be generous. Then

$$F_\phi := \bigoplus_{b\in B(G)} \bigoplus_{\pi\in\Pi_\phi(G_b)} i_{b!}^{\mathrm{ren}}\pi^{\oplus \dim \tau_\phi(b,\pi)}.$$

is a perverse Hecke eigensheaf w/ eigenvalue ϕ.

Rmks (1) Conj 1 + compatability with classical LLC with duality
$\Rightarrow$ Conj 2.

(2) Conj 1 + Conj 2 $\Rightarrow$ Conj 3.

(3) Conj 1 tells you how to compute $T_V i_{b!}^{\mathrm{ren}}\pi$
$\Rightarrow$ Conj 1 implies the Kottwitz Conj.

What's known?

(1) ϕ supercusp $\Rightarrow S_\phi = S_\phi^\natural$, $B(G) - B(G)_{\mathrm{basic}}$ doesn't contribute.

(2) All 3 conjs ok for GL_n & unram $U_{2n+1}/\mathbb{Q}_p$.

(3) Conj 2 & 3 ok for GSp_4.

Beyond ϕ supercusp:

__Hamann__ Assume G q-split, ϕ total & generic.
Then Conj 2 & 3 are true.
If $G = GL_n$, Hamann-Hansen $\Rightarrow$ Conj 1 true.

Assume G split, ϕ is the __trivial__ L-parameter.
Then $S_\phi = \hat{G}$, so $\mathrm{Irr}\, S_\phi = X^*(\hat{T})^+ = X_*(T)^+ \ni \lambda$

$$\text{BM-O bij}: \quad (\mathrm{triv}, \lambda) \longleftrightarrow (\underset{\lambda(\varpi)}{\overset{\|}{b_\lambda}}, \underset{i_B^{G_{b_\lambda}}(\mathbb{1})}{\overset{\|}{\pi_\lambda}})$$

$$
\begin{tikzcd}
q^{-1}(X_{triv}) = \mathcal{N}/\hat{G} \arrow[dr, bend right, hook, "\iota"'] & \tilde{\mathcal{N}}/\hat{G} \simeq \hat{\mathfrak{u}}/\hat{B} \arrow[l, "\pi"'] \arrow[r, "\eta"] & \pi/\hat{B} \\
& \mathrm{Par}_G . & \mathcal{L}_\lambda \arrow[u, squiggly]
\end{tikzcd}
$$

$A_\lambda := \pi_* \eta^* \mathcal{L}_\lambda \in \mathrm{Coh}(\mathcal{N}/\hat{G})$ Anderson-Jantzen sheaves.

<u>Conj</u>

$$
\begin{tikzcd}
\iota_* A_\lambda * i_{1!} W_\psi \simeq i^{ren}_{b_\lambda,!} \pi_\lambda \\
\iota_* A_{w_0(\lambda)} * i_{1!} W_\psi \simeq i^{ren}_{b_\lambda,\#} \pi_\lambda \arrow[u, dashed]
\end{tikzcd}
$$

Forced on you by expected compatibility of cat'cal LLC with duality and Eisenstein series.