A refined LLC for disconnected groups

Tasho Kaletha

§1 Connected case

F char 0, G/F conn red. $\rightsquigarrow \hat{G}$.

$\Pi(G)$ = irred adm $\mathbb{C}$-rep'n

$\Gamma = Gal(\bar{F}/F)$, W_F = Weil gp. $L_F = W_F$ or $W_F \times SL_2(\mathbb{C})$.

$\rightsquigarrow {}^LG = \hat{G} \rtimes \Gamma$.

(A) G quasi-split.

$$\Pi(G) \hookrightarrow \left\{ (\varphi, \rho) \;\middle|\; \begin{array}{l} \varphi : L_F \to {}^LG \\ \rho \in Irr(\pi_0(\bar{S}_\varphi)) \end{array} \right\} / \hat{G}\text{-conj}$$

where $S_\varphi = Cent(\varphi, \hat{G})$, $\bar{S}_\varphi = S_\varphi / Z(\hat{G})^\Gamma$.

(B) More general setting with G quasi-split.

$[\bar{z}] \in H^1(\Gamma, G/Z(G)) \rightsquigarrow G_{\bar{z}}$.

(a) Assume $[\bar{z}]$ lifts to $[z] \in H^1(\Gamma, G)$

(Vogan)
$$\Pi(G_{\bar{z}}) \hookrightarrow \left\{ (\varphi, \rho) \;\middle|\; \begin{array}{l} \varphi : L_F \to {}^LG \\ \rho \in Irr(\pi_0(S_\varphi), [z]) \end{array} \right\} / \hat{G}.$$

(Kneser-Kottwitz) $H^1(\Gamma, G) \to \pi_0(Z(\hat{G})^\Gamma)^*$.

(b) Drop assumptions; use Galois gerbe.

(Langlands-Kottwitz)

$$1 \to \mathcal{U} \to \mathcal{E} \to \Gamma \to 1$$

$$\rightsquigarrow H^1(\Gamma, G) \hookrightarrow H^1(\mathcal{E}, Z(G) \to G) \to H^1(\Gamma, G/Z(G))$$
$$[z] \longmapsto [\bar{z}]$$

Want (i) the latter is surjective

(ii) $H^1(\mathcal{E}, Z(G) \to G) \longrightarrow \pi_0(Z(\hat{\bar{G}})^+)^*$.

$$\begin{tikzcd}
\tilde{\hat{G}} \arrow[r] & \hat{G} & \text{univ cover} \\
S_\varphi^+ \arrow[u, phantom, "\cup" rotate=90] & S_\varphi \arrow[u, phantom, "\cup" rotate=90] \\
Z(\tilde{\hat{G}})^+ \arrow[r] \arrow[u, phantom, "\cup" rotate=90] & Z(\hat{G})^+ \arrow[u, phantom, "\cup" rotate=90]
\end{tikzcd}$$

$\parallel$ the preimage of $Z(\hat{G})^+$.

Conj $\quad \Pi(G_{\bar{z}}) \hookrightarrow \left\{ (\varphi, \rho) \;\middle|\; \begin{array}{l} \varphi : L_F \longrightarrow {}^LG \\ \rho \in \mathrm{Irr}(\pi_0(S_\varphi^+), [z]) \end{array} \right\} / \hat{G}\text{-conj}.$

(c) <u>char id</u> $\quad \varphi : L_F \longrightarrow {}^LG$ tempered. $\quad s \in S_\varphi$,

$\tilde{H} = \mathrm{Cent}(s, \hat{G})^\circ$

$\rightsquigarrow H$ quasi-split $/F$.

$$\begin{tikzcd}
L_F \arrow[d, dashed] \arrow[dd, bend right] & {}^LG \\
{}^LH \arrow[ur, dashed] \arrow[d, phantom, "\cong" rotate=90] & \\
{}^LH_I \arrow[uur, bend right] &
\end{tikzcd}$$

ignore the double cover by identifying ${}^LH$ & ${}^LH_I$.

Have $\quad \displaystyle\sum_{\pi \in \Pi_\varphi(G)} \mathrm{tr}(\rho_\pi(s)) \cdot \Theta_\pi(f) = \sum_{\pi \in \Pi_\varphi(H)} \dim(\rho_\pi) \cdot \Theta_\pi(f^H). \quad$ (char id)

$f^H \in \mathcal{C}_c^\infty(H(F))$,

(the map $f \longmapsto f^H$ of test func is called the transfer.)

$f \in \mathcal{C}_c^\infty(G_{\bar{z}}(F))$

<u>Twisted endoscopy</u>

G conn red $/F$

$G \curvearrowleft \theta$ F-automorphism

- Conj $\rightsquigarrow$ twisted conj

$g\gamma g^{-1} \qquad g\gamma\theta(g)^{-1}$.

- character $\rightsquigarrow$ twisted character

"$\mathrm{tr}\, \pi(g)$" $\qquad \mathrm{tr}(\pi(g) \circ I_\theta)$

with $I_\theta : \pi^\theta \xrightarrow{\sim} \pi$

disconnectedness : $G \curvearrowright G \cdot \theta \subset G \rtimes \langle \theta \rangle$.

Problems (1) Normalizations : (a) Choice of I_0

(b) transfer.

(2) Non-cyclic comp grps

(3) Language.

§2 Disconnected case

Assumption (1) $\tilde{G}$ affine F-alg gp, $G = \tilde{G}^\circ$ reductive.

(2) $\tilde{G}_{\bar{F}} \cong G_{\bar{F}} \rtimes \underset{\substack{\uparrow \\ \text{finite}}}{A}$, $A \curvearrowright G_{\bar{F}}$ in a pinned way.

Rmk If G adjoint, then (2) is automatic (but not otherwise).

Eg. $O(n)$ fine, $(M \rtimes W_G(M))_{\text{twisted}}$ also okay.

But normalizer of torus in SL_2 is not okay!

Classification · "split" $G \rtimes A$ / F split

· "forms" $H^1(\Gamma, \mathrm{Aut}(G \rtimes A))$

(i) $G/Z(G)^A \hookrightarrow \mathrm{Aut}(G \rtimes A)$ "inner forms".

(ii) $\mathrm{Aut}_{pin}(G \rtimes A)$: perverse pinning of G and $1 \rtimes A$ "G-split".

(iii) $Z^1(A, Z(G)) \hookrightarrow \mathrm{Aut}(G \rtimes A)$ "translation forms"

$z \longmapsto (g \rtimes a \mapsto z(a) \cdot g \rtimes a)$

(i) & (iii) commutes, (i) $\cap$ (iii) $= Z(G)/Z(G)^A = B^1(A, Z(G))$,

(i) · (iii) $\rtimes$ (ii) $= \mathrm{Aut}(G \rtimes A)$.

Today Quasi-split + inner.

$\tilde{G} = G \rtimes A$ quasi-split,

(G q-split & $A \curvearrowright G$ fix pinning.)

$[\bar{z}] \in H^1(\Gamma, G/Z(G)^A) \rightsquigarrow G_{\bar{z}}$.

Rational pts $\tilde{G}(F) = G(F) \rtimes A$,

$$1 \to G_{\bar{z}}(F) \to \tilde{G}_{\bar{z}}(F) \to \begin{array}{c} A \\ \cup \\ A^{[\bar{z}]} \end{array} \to 1.$$

Goal Classify $\Pi(\tilde{G}_{\bar{z}}(F))$, char id.

Postulate L-parameter for $\tilde{G}_{\bar{z}}$ are the same as for $G_{\bar{z}}$.

$$\varphi : L_F \longrightarrow {}^L G.$$

Define Parametrize $\Pi_\varphi(\tilde{G}_{\bar{z}})$ in terms of $\tilde{G}$, W_F.

Formalize char id.

$$\left(\begin{array}{l} \tilde{G} = A, \quad G = \{1\}. \\ \mathrm{Irr}(A) \xleftrightarrow{\;?\;} \text{arithmetic of } F \end{array} \right).$$

Recall $\varphi_1, \varphi_2 : L_F \to {}^L G$ are $G_{\bar{z}}$-equiv $\iff$ $\hat{G}$-conj.

$\rightsquigarrow$ S_φ = grp of self-equiv of φ.

Postulate $\varphi_1, \varphi_2 : L_F \to {}^L G$ are $\tilde{G}_{\bar{z}}$-equiv

$\iff$ φ_1, φ_2 are $\hat{G} \rtimes A^{[\bar{z}]}$-conj.

Reasonably $\tilde{\pi}|_{G_{\bar{z}}(F)} \cap \Pi_{\varphi_1}(G_{\bar{z}}) \neq \emptyset \iff \tilde{\pi}|_{G_{\bar{z}}(F)} \cap \Pi_{\varphi_2}(G_{\bar{z}}) \neq \emptyset$.

functorial property of LLC for $G_{\bar{z}}$.

$\rightsquigarrow$ $\tilde{S}_\varphi^{[\bar{z}]}$:= self-equiv classes of φ = $\mathrm{Cent}(\varphi, \hat{G} \rtimes A^{[\bar{z}]})$.

Rmk This indeed depends on $\bar{z}$ only.

Conj $\tilde{G}$ quasi-split: $\Pi_\varphi(\tilde{G}) \hookrightarrow \mathrm{Irr}(\pi_0(\tilde{S}_\varphi) \mid Z(\hat{\tilde{G}})^\Gamma)$

$\tilde{G}_{\bar{z}}$: lift $[\bar{z}]$ to $[z] \in H^1(\mathcal{E}, Z(\hat{G})^\wedge \to G)$

$$\Pi_\varphi(\tilde{G}_E) \hookrightarrow \mathrm{Irr}(\pi_0(\tilde{S}_\varphi^{[\bar{z}],+}), [z]).$$

Solves: (i) Choice of I_θ

(ii) Interaction of components.

char id $\tilde{s} \in \tilde{S}_\varphi^{[\bar{z}],+}$, $\hat{\tilde{H}} = \mathrm{Cent}(\tilde{s}, \hat{\tilde{G}})^\circ$, H quasi-split.

Rmk $[z]$ normalizes transfer.

Conj Characters identies hold as in the conn case.