Weight part of Serre's conjecture and Emerton-Gee stack

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§1 Geometric Breuil-Mezard

p prime, $n \geq 2$, $K/\mathbb{Q}_p$ finite with res field k.

$E/\mathbb{Q}_p \rightsquigarrow \mathcal{O}$, $\mathbb{F} = \mathcal{O}/(\varpi)$.

Emerton-Gee stack $\mathfrak{X}/\mathcal{O}$

"moduli" of n-dim'l p-adic repr $/G_K$."

Thm (Emerton-Gee)

(1) $\bar{\mathfrak{X}}_{red}$ equidim'l of dim $[K:\mathbb{Q}_p]\cdot\frac{n(n-1)}{2}$.

(2) $\mathrm{IrrComp}(\bar{\mathfrak{X}}_{red}) \xleftrightarrow{1-1}$ (mod p Serre weights of $GL_n(k)$)

$= \{$ irred mod p reprs of $GL_n(k)\}$.

For any Serre wt σ, get $\mathfrak{X}_\sigma \subset \bar{\mathfrak{X}}_{red}$.

$\rightsquigarrow$ $Z(\bar{\mathfrak{X}}_{red})$ = free abel gp on $\mathfrak{X}_\sigma$.

E.g. $K = \mathbb{Q}_p$, $\sigma = F(\lambda_1, \ldots, \lambda_n)$, $0 \leq \lambda_i - \lambda_{i+1} \leq p-1$

$\exists$ Zariski open in $\mathfrak{X}_\sigma$ where

maxl non-split

$$\bar{\rho} = \begin{pmatrix} \chi_1\omega_1^{\lambda_1+n-1} & & & \boxed{*} \\ & \chi_2\omega_2^{\lambda_2+n-1} & & \\ & & \ddots & \\ & & & \chi_n\omega_n^{\lambda_n} \end{pmatrix}$$

in which χ_i unramified & ω mod p cyclotomic.

Fix $\mu \in (\mathbb{Z}^n_+)^{\mathrm{Hom}(K, \bar{\mathbb{Q}}_\ell)}$ regular Hodge type.

τ inertia type.

(EG) $\mathcal{X}^{\mu,\tau} \subset \mathcal{X}$ is equidim'l of dim $[K:\mathbb{Q}_p] \frac{n(n-1)}{2}$.

$Z_{\mu,\tau} = Z(\bar{\mathcal{X}}^{\mu,\tau}) \in Z(\bar{\mathcal{X}}_{red})$ as cycles

$(\mu,\tau) \rightsquigarrow V(\mu,\tau)/E$ locally alg repr.

$\circlearrowleft$ $GL_n(\mathcal{O}_K)$

<u>Example</u> $K = \mathbb{Q}_p$.

- $\mu = (n-1, n-2, \cdots, 0)$, τ regular tame type.

$\rightsquigarrow V(\mu,\tau) = \sigma(\tau)$ Deligne-Lusztig repr.

- τ trivial,

$\rightsquigarrow V(\mu,\tau) = W(\mu - (n-1, n-2, \cdots, 0))$ Weil mod.

<u>Breuil-Mezard conj</u> For each Serre wt σ,

$\exists$ an (effective) cycle $Z_\sigma \in Z(\bar{\mathcal{X}}_{red})$

s.t. for all (μ,τ),

$$Z_{\mu,\tau} = \sum_\sigma \underbrace{m_\sigma(\mu,\tau)}_{\text{multi of } \sigma \text{ in } V(\mu,\tau)} \cdot Z_\sigma$$

<u>Known cases</u> - $n=2$, $K = \mathbb{Q}_p$ (Kisin, Paskunas, ...)

- $n=2$, $K/\mathbb{Q}_p$, $\underbrace{\text{potentially Barsotti-Tate}}_{\text{corr to smallest HT wts } (=\{0,1\})}$ (Gee-Kisin).

<u>Cycles</u> (Caraiani-Emerton-Gee-Savitt)

- $Z_\sigma = \mathcal{X}_\sigma$ if σ not twist of Steinberg

- $Z_{St \otimes \psi} = \mathcal{X}_{St\otimes\psi} + \mathcal{X}_\psi$.

- $n>2$, $K/\mathbb{Q}_p$ unram, μ small, τ tame & suff generic (LLLM).
- $n>2$, $K/\mathbb{Q}_p$ ramified, μ small, τ trivial (Bartlet?)

"Thm" $n=3$, $K=\mathbb{Q}_p$, $p\gg 0$, $\mu=(2,1,0)$, τ tame.
Then BD conj holds with

$$Z_{F(\lambda)} = \begin{cases} \chi_{F(\lambda)}, & \lambda_1-\lambda_2, \lambda_2-\lambda_3 < p-1 \\ \chi_{F(\lambda)} + \chi_{F(\lambda_2,\lambda_2,\lambda_3)}, & \lambda_1-\lambda_2 = p-1, \lambda_2-\lambda_3 < p-1 \\ \chi_{F(\lambda)} + \chi_{F(\lambda_1,\lambda_2,\lambda_2)}, & \lambda_1-\lambda_2 < p-1, \lambda_2-\lambda_3 = p-1. \\ \chi_{F(\lambda)} + \chi_{F(\lambda_2,\lambda_2,\lambda_3)} + \chi_{F(\lambda_1,\lambda_2,\lambda_2)} & \\ \quad + \chi_{F(\lambda_2,\lambda_2,\lambda_2)}, & \lambda_1-\lambda_2 = \lambda_2-\lambda_3 = p-1. \end{cases}$$

Rmk If $n>3$, $Z_{F(\lambda)}$ is not irred.

§2 Weight part of Serre Conj

$F/\mathbb{Q}$ tot real, p inert, $K=F_p$.

$\bar{r}: G_F \longrightarrow GL_n(\bar{\mathbb{F}}_p)$ conti irrep, $\bar{r}|_{G_{F}} = \bar{\rho}$.

Alg modular form for definite unitary group.

Def'n $\bar{r}$ modular of Serre wt σ if

$$S(K^p, \sigma)_{m_{\bar{r}}} \neq 0.$$

Wt part Serre conj

It predicts $W(\bar{r})$ (Serre wts of $\bar{r}$) in terms of $\bar{\rho}$.

E.g. $n=2$, $W_{BDJ}(\bar{\rho})$

$n>2$, $\bar{\rho}$ tame & generic, $W^?(\bar{\rho})$ (Herzig).

Conj Assume BM conj with effective cycles Z_σ.

Define $W_{BM}(\bar{\rho}) = \{\sigma : \bar{\rho} \in Z_\sigma\}$

Then (1) If $\bar{r}$ modular, $W(\bar{r}) = W_{BM}(\bar{\rho})$.

(2) $W_{BM}(\sigma) \supset W^g(\bar{\rho}) = \{\sigma : \bar{\rho} \in \mathcal{X}_\sigma\}$.

Thm (LLLM, 2022) $\bar{r}$ irred modular & Taylor-Wiles hypothesis.

Assume $\bar{\rho}$ tame, suff generic. Then

$$W(\bar{r}) = W_{BM}(\bar{\rho}) \overset{?}{=} W(\bar{\rho}).$$

"Thm" $n=3$, Same $\bar{r}$ as above, $p \gg 0$. $\bar{\rho}$ ss, $K = \mathbb{Q}_p$.

Then $W(\bar{r}) = W_{BM}(\bar{\rho})$.

§3 Local models

$K = \mathbb{Q}_p$, $\mu = (n-1, n-2, \cdots, 0)$

Input geometry:

- $\mathcal{X}^{\mu,\tau}$ to TW method.
- $M(\mu)/\mathcal{O}$ Pappas-Zhu local model with Iwahori level.
- τ suff generic. $\mathcal{X}^{\mu,\tau} \underset{\text{sm top}}{\sim} M(\mu)^{\nabla_\tau} \subset M(\mu)$.

Naive hope $M(\mu)^{\nabla_\tau}$ normal (but false for $n \geq 3$).

Thm (LLLM) $M(\mu)^{\nabla_\tau}$ is unibranch at T-fixed points.

The case $n=3$ & non-generic:

Rough strategy

$$\begin{tikzcd} (\rho, \mathfrak{M}) \in \mathcal{Y}^{\mu,\tau} \arrow[r] \arrow[d, mapsto] & \mathcal{X}^{\mu,\tau} \\ \mathfrak{M} \in \mathcal{Y}^{\mu,\tau}_{\mathrm{Kis}} & \end{tikzcd}$$

partial resolution

BK mod

Key (1) $\mathcal{Y}^{\mu,\tau}_{\mathrm{Kis}} \sim \widetilde{M}(\mu)^{\nabla_\tau'}$ in some sense

(2) $\mathcal{Y}^{\mu,\tau} \longrightarrow \mathcal{Y}^{\mu,\tau}_{\mathrm{Kis}}$ explicit blow-up.

"Thm" $p \gg 0$, $n = 3$, $K = \mathbb{Q}_p$.

$\mathcal{Y}^{\mu,\tau}$ is normal (away from 0-dim'l locus.)