Representation theory via 6-functor formalism

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§1 6-functor formalism

Fix a geometric setting.

$\mathcal{C}$ = Cat of geom objects

E = Collections of edges in $\mathcal{C}$ stable under comp & pullback.

<u>Def'n</u> $\mathrm{Corr}(\mathcal{C}, E)$ with

objects: Ob $\mathcal{C}$ "modification from Y to X"

morphs: $\mathrm{Hom}(Y, X) = \{ Y \xleftarrow{g} Y' \xrightarrow{f} X : f \in E \}$

under compositions

$$\begin{tikzcd} Z' \times_Y X' \arrow[r] \arrow[d] & Y' \arrow[r] \arrow[d] & X \\ Z' \arrow[r] \arrow[d] & Y & \\ Z & & \end{tikzcd}$$

There is a $\otimes$-functor on $\mathrm{Corr}(\mathcal{C}, E)$ via $X \otimes Y := X \times Y$.

<u>Def'n</u> A <u>3-functor formalism</u> is a lax $\otimes$-functor

$$D : (\mathrm{Corr}(\mathcal{C}, E), \otimes) \longrightarrow (\mathrm{Cat}, \times)$$

$\rightsquigarrow \forall X \in \mathcal{C}$, $\forall$ $Y \xleftarrow{g} Z \xrightarrow{f} X$.

$D(X)$ = "sheaves on X" w/ $\otimes$ on $D(X)$,

$D(Y \xleftarrow{g} Z \xrightarrow{f} X) = D(Y) \xrightarrow{g^*} D(Z) \xleftarrow{f_!} D(X)$.

Encodes projection formula + proper base change.

<u>Def</u> $\mathcal{D}$ is a <u>6-functor formalism</u> if $\otimes, f^*, f_!$
have right adjoints $\underline{\mathrm{Hom}}$, $f_*, f^!$.

<u>Examples</u> (a) $\mathcal{C} = \{$v-stacks on Perf$\}$, ℓ prime, $\Lambda = \mathbb{F}_\ell$-alg
$\rightsquigarrow$ get 6-functor formalism $D(-,\Lambda)$ on $\mathcal{C}$
with $E = \{\ell$-fine maps$\}$

<u>note</u> (map of rigid vars) $\in E$
(map of classifying stacks of p-adic Lie gps) $\in E$.

- $\ell \neq p$: $D(-,\Lambda) = D_{\acute{e}t}(-,\Lambda)$
- $\ell = p$: $D(-,\mathbb{F}_p) = D^a_\square(-, \mathcal{O}^+_x/\pi)^\varphi \supseteq D_{\acute{e}t}(-,\mathbb{F}_p)^{\omega}$.

(b) $\mathcal{C} = \{$stacks on profinite sets$\}$, Λ any ring.
$\rightsquigarrow D(X,\Lambda) = \{\Lambda$-valued sheaves on $X\}$
$E = \{\Lambda$-fine maps$\}$.

<u>Example</u> $H \to G$ of p-adic Lie gps
with associated map $f: */H \longrightarrow */G$.

Then $D(*/G, \Lambda) = D^{sm}(G,\Lambda)$, and

$f^*: D^{sm}(G,\Lambda) \longrightarrow D^{sm}(H,\Lambda)$ restr / infla
$f_*: D^{sm}(H,\Lambda) \longrightarrow D^{sm}(G,\Lambda)$ Ind / cohomology
$f_!: D^{sm}(H,\Lambda) \longrightarrow D^{sm}(G,\Lambda)$ c-ind / $\approx$ homology.

§2 Admissibility and coadmissibility

From now on, assume $E = \{\text{all maps of } \mathcal{C}\}$.

$\rightsquigarrow$ $\mathrm{Corr}(\mathcal{C}) := \mathrm{Corr}(\mathcal{C}, E)$ for simplicity.

Note that $\mathrm{Corr}(\mathcal{C})$ is enriched over itself.

& $\underline{\mathrm{Hom}}_{\mathcal{C}}(X, Y) = X \times_{\mathcal{C}} Y$.

Def Transfering the enrichment along D gives the 2-cat of kernels K_D:

- objs: $\mathrm{Ob}\,\mathcal{C}$
- morphs: $\mathrm{Hom}_{K_D}(X, Y) = D(X \times Y)$.

Thm (Liu-Zheng, Fargues-Scholze)

There are natural functors of 2-cats:

$$\begin{array}{ccccccc} \mathcal{C} & \longrightarrow & \mathrm{Corr}(\mathcal{C}) & \longrightarrow & K_D & \longrightarrow & \mathrm{Cat} \\ X & \longmapsto & X & \longmapsto & X & \longmapsto & D(X) \\ & & & & A \in D(X \times Y) & \longmapsto & \pi_{2,!}(A \otimes \pi_1^*(-)) \end{array}$$

For $S \in \mathcal{C}$, denote $K_{D,S}$ the version where we start with $\mathcal{C}_{/S}$.

Def Let $f: X \to S$, $M \in D(X)$.

(a) M is <u>f-admissible</u> if it is left adjoint when viewed in $\mathrm{Hom}_{K_{D,S}}(X, S) = D(X)$.

The associated right adj is denoted $SD_f(M) \in D(X)$.

(b) M is <u>f-coadmissible</u> if it is right adjoint when viewed in $\mathrm{Hom}_{K_{D,S}}(X, S) = D(X)$.

The associated left adj is denoted $PD_f(M) \in D(X)$.

Rmk $K_{D,S}^{op} \simeq K_{D,S}$.

$$\Rightarrow \left(\begin{array}{l} M \text{ f-adm} \Leftrightarrow SD_f(M) \text{ f-adm } \& \ SD_f(SD_f(M)) = M. \\ M \text{ f-coadm} \Leftrightarrow SD_f(M) \text{ f-coadm } \& \ PD_f(PD_f(M)) = M. \end{array} \right)$$

Def Let $g : Y \to X$ in $\mathcal{C}$.

- g is D-smooth if $1 \in D(Y)$ is g-adm & $SD(1)$ are inv.
  $$\Rightarrow g^! = g^* \otimes \boxed{SD_g(1)} =: \omega_g.$$
- g is D-proper if $1 \in D(Y)$ is g-coadm & $PD(1)$ are inv.
  $$\Rightarrow g_!(- \otimes \underbrace{\boxed{PD_g(1)}}_{\mathcal{D}_g}) = g_*(-).$$

Thm In Example (a),

let f be a map of analytic adic spaces $/ \mathbb{Q}_p$.

(i) f sm of pure dim d

$\Rightarrow f$ D-sm and $\omega_f = 1[2d](d)$.

(ii) f proper $\Rightarrow f$ D-proper, $\mathcal{D}_f = 1$.

Thm In Example (b),

let G p-adic Lie grp with trivial H

& let $f : */G \to *$.

(i) If $\Lambda \in \mathrm{Alg}_{\mathbb{F}_p}$ or $\mathrm{Alg}_{\mathbb{Z}[\frac{1}{p}]}$,

then f is D-smooth w/

$$\omega_G := \omega_f = \left\{ \begin{array}{l} \text{Concentrated in deg } 0, \\ \quad \text{parametrizing Haar measures} / \Lambda / \mathbb{Z}[\tfrac{1}{p}] \\ \left(\begin{array}{l} \text{Concentrated in deg } -\dim G, \\ \quad \text{parametrizing Haar measures} / \Lambda / \mathbb{F}_p, \end{array} \text{resp.} \right) \end{array} \right\}$$

(ii) If G pro-p p-torsion-free,

then f is D-proper, $D_f = 1$.

<u>Applications</u> G p-adic Lie gp.

① $V \in D(*/G, \Lambda)$ is admissible

$\Leftrightarrow$ $\forall$ pro-p p-torsion-free open subgps $K \subset G$,

$V^K \in D(\Lambda)$ perfect,

and $SD(V) = \underline{\mathrm{Hom}}(V, \omega_G)$.

<u>Thm</u> (Hansen–Mo) $G = GL_n(F)$, D central div alg of inv $\frac{1}{n}/F$.

Λ $\mathbb{F}_p$-alg.

Consider

$$\begin{tikzcd} & \Omega^{n-1}/G = \mathcal{M}_\infty / G \times D^\times = \mathbb{P}^{n-1}/D^\times \arrow[dl, "f"', "D\text{-sm}"] \arrow[dr, "g", "{D\text{-sm}, D\text{-proper}}"'] & \\ */G & & */D \end{tikzcd}$$

$\rightsquigarrow$ $JL := g_* f^* : D(*/G, \Lambda) \longrightarrow D(*/D^\times, \Lambda)$.

preserves admissibility, and

$$JL(V)^\vee = JL(V^\vee)[2n-2](n-1).$$

② Example (b).

<u>Prop</u> (i) $V \in D(*/G, \Lambda)$ is coadm $\Leftrightarrow$ V cpt.

(ii) $\mathbb{D}_{BZ} := P\mathbb{D} : D(*/G, \Lambda)^{\omega, op} \xrightarrow{\sim} D(*/G, \Lambda)^{\omega}$.

(iii) $D_{BZ}(c\text{-}\mathrm{Ind}_K^G V) = c\text{-}\mathrm{Ind}_K^G V^\vee$. V cpt K-rep.

(iv) $D_{BZ}(M) = \mathrm{Hom}_G(M, C_c(G, \Lambda))$.

<u>Cor</u> $I \subseteq G$ pro-p Iwahori. $H_I = \mathrm{End}(c\text{-}\mathrm{Ind}_I^G \mathbb{1}) \in \mathrm{Alg}(D(\Lambda))$.

$\Rightarrow$ D_{BZ} induces $H_I \xrightarrow{\sim} H_I^{op}$.

③ G reductive. $P = MU \subseteq G$ parabolic, opposite $\bar{P}$.

$$\begin{tikzcd} & */M & \\ */P \arrow[ur, "f"] \arrow[dr, "g"'] & & */\bar{P} \arrow[ul, "\bar{f}"'] \arrow[dl, "\bar{g}"] \\ & */G & \end{tikzcd}$$

Let $i_P^G := g_* f^*$ = parabolic induction

$r_P^G := f_* g^!$ = right adj

$l_P^G := f_! g^*$ ≈ left adj.

<u>Thm</u> (Mann-Heyer-Zou, in process)

(i) $\exists$ natural map $r_{\bar{P}}^G \longrightarrow l_P^G$.

(ii) Let $\mathbb{Z} \curvearrowright G$ via conj by the cochar for P.

For every $G \rtimes \mathbb{Z}$-rep V,

$$(r_{\bar{P}}^G V)^{\mathbb{Z}} \xrightarrow{\sim} (l_P^G V)^{\mathbb{Z}}.$$

<u>Cor</u> V adm G-rep (and $l_P^G V$ adm when Λ is a field)

$\Rightarrow$ $r_{\bar{P}}^G V = l_P^G V$.

pf. by applying thm to $\prod_{\mathbb{Z}} V$. □