Duality for p-adic pro-étale cohomology of analytic curves

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(Joint with Pierre Colmez & Sally Gilles.)

$K/\mathbb{Q}_p$ finite. $\mathcal{O}_K \twoheadrightarrow k$, $\mathcal{G}_K = \mathrm{Gal}(\bar{K}/K)$, $C = \hat{\bar{K}}$.

§1 Arithmetic duality

<u>Thm</u> (CGN) X sm geom irred dagger curve var of dim $1/K$.

Then (1) $\exists$ natural trace map of solid $\mathbb{Q}_p$-v.s.

$$\mathrm{Tr}_X : H^4_{\mathrm{pro\acute{e}t}}(X, \mathbb{Q}_p(2)) \xrightarrow{\sim} \mathbb{Q}_p$$

(2) The pairing

$$H^i(X, \mathbb{Q}_p(j)) \otimes^{\blacksquare}_{\mathbb{Q}_p} H^{4-i}_c(X, \mathbb{Q}_p(2-j))$$
$$\to H^4_c(X, \mathbb{Q}_p(2)) \simeq \mathbb{Q}_p$$

is a perfect pairing

$$r_{X,i} : H^i(X, \mathbb{Q}_p(j)) \xrightarrow{\sim} H^{4-i}_c(X, \mathbb{Q}_p(2-j))^*$$
$$r^c_{X,i} : H^i_c(X, \mathbb{Q}_p(j)) \xrightarrow{\sim} H^{4-i}(X, \mathbb{Q}_p(2-j))^*.$$

<u>Rmks</u> (1) X Stein. $U_n \Subset U_{n+1}$ affinoid.

$$R\Gamma_c(X, \mathbb{Q}_p) := (R\Gamma(X, \mathbb{Q}_p) \to R\Gamma(\partial X, \mathbb{Q}_p))$$
$$R\Gamma(\partial X, \mathbb{Q}_p) := \mathrm{colim}_n\, R\Gamma(X \setminus U_n, \mathbb{Q}_p).$$

(2) X dim 1, X proper affinoid Stein

X proper : all cohom gp are finite $/\mathbb{Q}_p$

X Stein : $H^i(X, \mathbb{Q}_p)$ nuclear Frechet,

$H^i_c(X, \mathbb{Q}_p)$ cpt type.

X dagger affinoid : $H^i(X, \mathbb{Q}_p)$ cpt type,

$H^i_c(X, \mathbb{Q}_p)$ nuclear Frechet.

(3) X dim 1, partially proper,

$\exists$ derived duality in $D(\mathbb{Q}_{p,\square})$

$$r_X : R\Gamma(X, \mathbb{Q}_p(j)) \xrightarrow{\sim} \mathbb{D}(R\Gamma_c(X, \mathbb{Q}_p(2-j))[4]),$$

$$\mathbb{D} = \underline{R\mathrm{Hom}}(-, \mathbb{Q}_p)$$

- $r_{X,i} \Leftarrow \mathrm{Ext}^i(H_c^j, \mathbb{Q}_p) = 0, \ \forall i > 1$
- $r_{X,i}^c \Leftarrow H_c^i$ reflexive.

(4) Conj X sm Stein $/K$, geom irred, dim $= d$.

Then (i) $H^i(X, \mathbb{Q}_p)$, $H_c^i(X, \mathbb{Q}_p)$

nuclear Frechet, cpt type.

(ii) We have isom

$$R\Gamma(X, \mathbb{Q}_p(j)) \xrightarrow{\sim} \mathbb{D}(R\Gamma_c(X, \mathbb{Q}_p(d+1-j))[2d+2]).$$

(5) E.g. $X = D$ open unit disc

$$H^1(X, \mathbb{Q}_p(1)) \simeq \mathcal{O}(D)/K \oplus H^1(\mathcal{G}_K, \mathbb{Q}_p(1))$$

$$H_c^3(X, \mathbb{Q}_p(1)) \simeq \mathcal{O}(\partial D)/\mathcal{O}(D) \oplus H^1(\mathcal{G}_K, \mathbb{Q}_p)$$

"ghost circle", proper of "dim $\frac{1}{2}$".

Duality by : Galois duality + $\underbrace{\text{coherent duality}}$.

$$H^i(\mathcal{G}_K, \mathbb{Q}_p) \simeq H^{2-i}(\mathcal{G}_K, \mathbb{Q}_p(1))^*$$

$$\left(\begin{array}{l} H^0(D, \Omega_D^1) \simeq H^1(D, \mathcal{O}_D)^* \\ \text{have } \mathcal{O}(D)/K \simeq H^0(D, \Omega_D^1) \\ \mathcal{O}(\partial D)/\mathcal{O}(D) \simeq H_c^1(D, \mathcal{O}_D). \end{array} \right)$$

(6) Solid v.s. classical functional:

Had to use solid math:

(i) use derived dual $\mathbb{D}$

(ii) topological Hochschild–Serre spectral seq'ce.

§2 Geometric duality

Conj (Verdier duality) X sm Stein $/C$

Then $\exists$ natural isom

$$R\Gamma(X, \mathbb{Q}_p(j)) \xrightarrow{\sim} R\mathrm{Hom}_{VS}(R\Gamma_c(X, \mathbb{Q}_p(d+1-j))[2d], \mathbb{Q}_p(1)).$$

where VS = v-sheaves of $\mathbb{Q}_p$-v.s.

E.g. $X = D$, nonzero cohom:

$$H^0(D, \mathbb{Q}_p(j)) = \mathbb{Q}_p(j),$$
$$H^1(D, \mathbb{Q}_p(j)) = (\mathcal{O}(D)/C)(j-1),$$
$$H^2_c(D, \mathbb{Q}_p(j)) \simeq \mathbb{Q}_p(j-1) \oplus \frac{\mathcal{O}(\partial D)}{\mathcal{O}(D)}(j-1).$$

See: $\mathbb{Q}_p$-v.s. + coherent duality,

No Poincaré duality.

But pass to VS,

- Anschütz-Le Bras:

$$\mathrm{Hom}_{VS}(\mathbb{Q}_p, \mathbb{Q}_p(1)) \cong \mathbb{Q}_p(1),$$
$$\mathrm{Ext}^1_{VS}(\mathbb{Q}_p, \mathbb{Q}_p(1)) = 0,$$
$$\mathrm{Hom}_{VS}(\mathbb{G}_a, \mathbb{Q}_p(1)) = 0,$$
$$\mathrm{Ext}^1_{VS}(\mathbb{G}_a, \mathbb{Q}_p(1)) = C$$

$$0 \to \mathbb{Q}_p(1) \to B_{cris}^{+, \varphi=p} \to \mathbb{G}_a \to 0.$$

- $\mathrm{Ext}^i_{VS} = 0$ with $i \geq 2$.

Get $\mathrm{Ext}^1_{VS}(H^2_c(D, \mathbb{Q}_p(j)), \mathbb{Q}_p(1)) \simeq H^1(D, \mathbb{Q}_p(j))$.

$$0 \to \cdots \to H^1_c \to \cdots \to H^2_c(D, \mathbb{Q}_p(j)) \to \mathrm{Hom}_{VS}(H^0(D, \mathbb{Q}_p(2-j)), \mathbb{Q}_p) \to 0.$$

Proof of main thm

E.g. X proper sm, dim d.

pf: · geometric Poincaré duality by Zavyalov, Mann:

$$H^i(X_C, \mathbb{Q}_p(j)) \simeq H^{2d-i}(X_C, \mathbb{Q}_p(d-j))^*.$$

· HS spectral seq.

· Galois duality.

<u>Step 1</u> $d=1$, Stein.

Geometric comparison thm (AGN, CDN).

(i) Vanishings: $H^i(X_C, \mathbb{Q}_p) = 0, \quad i \neq 0,1.$

$$H^i_c(X_C, \mathbb{Q}_p) = 0, \quad i \neq 1,2$$

(ii) isom: $H^0(X_C, \mathbb{Q}_p) = \mathbb{Q}_p$, Hyodo-Kato.

$$H^1_*(X, \mathbb{Q}_p(1)) = HK^1_*(X_C, 1)$$

$\varphi, N, \mathcal{G}_K$. φ, N

with $HK^j_*(X_C, i) := (H^j_{HK,*}(X_C) \hat{\otimes}^D \hat{B}^+_{st})^{\varphi = p^i, N=0}$

if $HK^j_*(X, i)$ finite (which is BC).

(iii) Exact seq

$$0 \longrightarrow \mathcal{O}(X_C)/C \longrightarrow H^1(X_C, \mathbb{Q}_p(1)) \longrightarrow HK^1(X_C, 1) \longrightarrow 0$$

$$HK^1_c(X_C, 2) \longrightarrow H^1 DR_c(X_C, 2) \longrightarrow H^2_c(X_C, \mathbb{Q}_p(2)) \xrightarrow{Tr_{X_C}} \mathbb{Q}_p(1) \longrightarrow 0.$$

$$DR_c(X_C, i) := (H^1_c(X, \mathcal{O}_X) \otimes^{\square}_K (B^+_{dR}/F^i) \to H^1_c(X, \Omega^1) \otimes^{\square}_K (B^+_{dR}/F^{i-1}))[-1]$$

Arithmetic trace

$$Tr_X : H^4_c(X, \mathbb{Q}_p(2)) \simeq H^2(\mathcal{G}_K, H^2_c(X_C, \mathbb{Q}_p(2)))$$

$$\downarrow$$

$$H^2(\mathcal{G}_K, \mathbb{Q}_p(1)) \simeq \mathbb{Q}_p$$

<u>Step 2</u> Galois descent.

$$E_2^{i,j} = H^i(\mathcal{G}_K, H^j_*(X_C, \mathbb{Q}_p(s))) \Rightarrow H^{i+j}_*(X, \mathbb{Q}_p(s)).$$

Galois coh: $H^0(X,\Omega) \otimes_K^{\square} C(s) \to H^1_c(X,\mathcal{O}) \otimes_K^{\square} C(s)$

$\longrightarrow HK^1(X_C, 1)(s)$

$H^i(\mathcal{G},\mathcal{O}) \rightsquigarrow$ $H^0(X,\Omega)$ $\quad H^1_c(X,\mathcal{O})$ $\quad H^1(\mathcal{G}_K, V)$ ass $H^1_{HK} < \infty$.

- assuming pro-étale product is compatible with coh product & Galois product.
- modulo ext'n issues:

  $H^1(X,\mathbb{Q}_p)$ - nuclear Fréchet $\rightsquigarrow H^0(X,\Omega)$

  H^1_c - cpt type.