A local Jacquet-Langlands correspondence for locally analytic D-modules

Juan Rodriguez Camargo

(Joint w/ Gabriel Dospinescu.)

$F/\mathbb{Q}_p$ finite, D/F quaternion alg.

<u>Goal</u> Study the functor of Scholze of p-adic JL in the realm of loc an reps.

Let C/F some alg closure.

$$\begin{tikzcd}
& GL_2(F)\times D^\times \circlearrowright \mathcal{M}_\infty \text{ (perf'd Drinfeld space).} \arrow[dl, "\pi_{GM}"'] \arrow[dr, "\pi_{HT}"] & \\
D^\times \circlearrowright \mathbb{P}^1_{LT,C} & & \Omega = \mathbb{P}^1_{Dr,C}\setminus\mathbb{P}^1(F) \circlearrowleft GL_2(F)
\end{tikzcd}$$

$GL_2(F)$-torsor $\longrightarrow \pi_{GM}$ $\qquad$ π_{HT} $D^\times$-torsor.

As diamonds, $\mathbb{P}^{1,\diamond}_{LT}/D^\times \simeq \Omega^\diamond/GL_2(F)$.

<u>Def</u> π adm unitary Banach rep'n of $GL_2(F)$.
Let $\mathcal{F}_\pi/\mathbb{P}^1_{LT}$ the associated pro-étale sheaf.

$$JL(\pi) := R\Gamma_{\text{proét}}(\mathbb{P}^1_{LT}, \mathcal{F}_\pi) \circlearrowleft D^\times.$$

<u>Thm</u> (Scholze) $JL(\pi)$ is an adm Banach rep'n of $D^\times$.
Moreover, $JL(\pi)\otimes_{\mathbb{Q}_p} C = R\Gamma_{\text{proét}}(\mathbb{P}^1_{LT}, \mathcal{F}_\pi \hat{\otimes} \hat{\mathcal{O}})$.

<u>Question</u> Is JL compatible with la rep?

<u>Answer</u> In this talk: affirmative ans from "two different directions".

<u>Some notations</u> · Lie $GL_2(F) = \mathfrak{g}_G$, Lie $D^\times = \mathfrak{g}_D$.

· $Z(\mathfrak{g}_G) \subset U(\mathfrak{g}_G)$, $Z(\mathfrak{g}_D) \subset U(\mathfrak{g}_D)$.

For $V \circlearrowleft H$, $V^{H\text{-}la}$ (or V^{la}) is the subsp of la rep'ns.

<u>Rmk</u> V Banach adm rep'n of H
$\Rightarrow V^{la}$ usual la vectors

π as before, $H^i(JL(\pi)^{la}) = JL^i(\pi)^{la}$.

<u>Thm</u> (Dospinescu - Rodriguez Camargo)
π as before. Then
$$JL(\pi)^{D^\times\text{-}la} = JL(\pi^{GL_2(F)\text{-}la}).$$
Moreover, $Z(\mathfrak{g}_D) \simeq Z(\mathfrak{g}_G)$ act in the same way.

<u>Thm</u> (Dospinescu - Rodriguez Camargo)
(1) $\widehat{\mathcal{O}}_{\mathcal{M}_\infty}$ structure sheaf on $\mathcal{M}_\infty$.
Then, as sheaves on $\mathcal{M}_\infty$, we have
$$\widehat{\mathcal{O}}_{\mathcal{M}_\infty}^{GL_2(F)\text{-}la} = \widehat{\mathcal{O}}_{\mathcal{M}_\infty}^{D^\times\text{-}la} =: \mathcal{O}_{\mathcal{M}_\infty}^{la} \text{ (lowest sheaf).}$$
Moreover, the action of $Z(\mathfrak{g}_G) = Z(\mathfrak{g}_D)$ are the same.
(2) For either sheaf $\mathbb{B}_I$ arising from FF curve,
$$\mathbb{B}_I^{GL_2(F)\text{-}la} = \mathbb{B}_I^{D^\times\text{-}la}$$
(actural derived sheaves).

To prove the previous thm, one needs two inputs:

Lem H cpt p-adic Lie gp,
V^+ p-adic complete rep'n of H.
Suppose $H \circlearrowright V^+/p^{\text{n}}$ factors through finite quotient.
Then $H \circlearrowright V^+[\frac{1}{p}]$ is locally analytic.

Thm (Zavyalov) Let $\mathcal{U} = \mathrm{Spa}(A, A^+)$ sm $/C$.
Then $R\Gamma_{\text{proét}}(\mathcal{U}, \hat{\mathcal{O}}^+)$ is almost coherent over A^+.

$\Rightarrow$ $H \circlearrowright \mathcal{U}$, then $H \circlearrowright R\Gamma_{\text{proét}}(\mathcal{U}, \hat{\mathcal{O}}^+/p)$
almost factor through finite quotient.
$\Rightarrow$ $H \circlearrowright R\Gamma_{\text{proét}}(\mathcal{U}, \hat{\mathcal{O}})$ is locally analytic
$H \circlearrowright R\Gamma_{\text{proét}}(\mathcal{U}, \mathbb{Q}_p)$ is also locally analytic

Q Does JL preserve la rep's?
A (Boring) No.
E.g. if π = alg rep'n then $JL(\pi)$ is a Banach-Colmez space.

Expectation $JL(\pi)$ should be a global section of an adm la rep of $D^\times$ over $X_{FF,C}$.

Some evidence by working over $\hat{\mathcal{O}}$:
Def (Pan) $\mathcal{O}^{la}_{\mathcal{M}_\infty} = \hat{\mathcal{O}}^{GL_2(F)\text{-}la}_{\mathcal{M}_\infty}$, $\hat{\mathcal{O}}^{D^\times\text{-}la}_{\mathcal{M}_\infty}$.
$\rightsquigarrow$ $\mathcal{M}^{la}_\infty = (|\mathcal{M}_\infty|, \mathcal{O}^{la}_{\mathcal{M}_\infty})$, $\mathcal{M}_\infty = (|\mathcal{M}_\infty|, \hat{\mathcal{O}}_{\mathcal{M}_\infty})$.

$$\begin{tikzcd} & \mathcal{M}_\infty^{la} \arrow[dl, "\pi_{GM}"'] \arrow[dr, "\pi_{HT}"] & \\ \mathbb{P}^1_{LT} & & \Omega \subseteq \mathbb{P}^1_{Dr}. \end{tikzcd}$$

Warning π_{GM} & π_{HT} are NOT torsors anymore!

LT side:

$\mathbb{P}^1_{LT} = (D^\times \otimes_{\mathbb{Q}_p} C)/B$

$\circlearrowleft D^\times$

$\alpha : \mathfrak{g}_D \otimes \mathcal{O}_{\mathbb{P}^1_{LT}} \twoheadrightarrow T_{\mathbb{P}^1_{LT}}$

$\mathfrak{g}_D \supseteq \bar{\mathfrak{b}}_0 \supseteq \bar{\mathfrak{n}}_0$

Drinfeld side:

$\mathbb{P}^1_{Dr} = GL_2/B$

$\circlearrowleft GL_2(F)$

$\beta : \mathfrak{g}_G \otimes \mathcal{O}_{\mathbb{P}^1_{Dr}} \twoheadrightarrow T_{\mathbb{P}^1_{Dr}}$

$\mathfrak{g}_G \supseteq \mathfrak{b}_0 \supseteq \mathfrak{n}_0$

Def LT side: $\mathcal{D}_{LT} = \mathcal{U}_{\mathcal{O}_{\mathbb{P}^1_{LT}}}(T_{\mathbb{P}^1_{LT}}) = \mathcal{U}_{\mathcal{O}_{\mathbb{P}^1_{LT}}}(\mathfrak{g}_D^\circ/\bar{\mathfrak{b}}^\circ)$

$\widetilde{\mathcal{D}}_{LT} = \mathcal{U}_{\mathcal{O}_{\mathbb{P}^1_{LT}}}(\mathfrak{g}_D^\circ/\bar{\mathfrak{n}}^\circ)$

all differential operators of line bundles.

Dr side: $\mathcal{D}_{Dr} = \mathcal{U}_{\mathcal{O}_{\mathbb{P}^1_{Dr}}}(T_{\mathbb{P}^1_{Dr}}) = \mathcal{U}_{\mathcal{O}_{\mathbb{P}^1_{Dr}}}(\mathfrak{g}_G^\circ/\mathfrak{b}^\circ)$.

$\widetilde{\mathcal{D}}_{Dr} = \mathcal{U}_{\mathcal{O}_{\mathbb{P}^1_{Dr}}}(\mathfrak{g}_G^\circ/\mathfrak{n}^\circ)$.

Def $D^\times \rtimes \widetilde{\mathcal{D}}_{LT} \subset \mathrm{Mod}(\mathbb{P}^1_{LT})$

space of solid la $D^\times$-equiv $\widetilde{\mathcal{D}}$-mod.

Thm (Pan-Rodriguez Camargo) $\mathfrak{n}^\circ \subset \mathfrak{g}_G^\circ$ & $\bar{\mathfrak{n}}^\circ \subset \mathfrak{g}_D^\circ$, w/ actions by $\mathcal{O}^{la}_{\mathcal{M}_\infty}$ trivially

$\Rightarrow \widetilde{\mathcal{D}}_{Dr}, \widetilde{\mathcal{D}}_{LT} \circlearrowright \mathcal{O}^{la}_{\mathcal{M}_\infty}$

Thm (Dospinescu-Rodriguez Camargo)

(1) $\exists$ functor

$$\Omega \subseteq \mathbb{P}^1_{Dr}$$

$$JL^{\tilde{D}} : GL_2(\mathbb{Q}_p)-\tilde{\mathcal{D}}_{Dr}-\mathrm{Mod}(\Omega) \longrightarrow D^\times-\tilde{\mathcal{D}}_{LT}-\mathrm{Mod}(\mathbb{P}^1_{LT}).$$

(2) π loc an rep'n of $GL_2(\mathbb{Q}_p)$

$\Rightarrow$ $JL(\pi \hat{\otimes} \hat{\mathcal{O}}) = R\Gamma_{an}(\mathbb{P}^1_{LT}, JL^{\tilde{D}}(\tilde{\mathcal{D}}_{Dr} \otimes_{U(\mathfrak{g})} \pi))$.

(3) $\mathcal{F}$ adm $GL_2(\mathbb{Q}_p) \times \tilde{\mathcal{D}}_{Dr}$-mod over Ω

$\Rightarrow$ $JL^{\tilde{D}}(\mathcal{F})$ is adm $D^\times$-$\mathcal{D}_{LT}$-mod

$\Rightarrow$ As $\mathbb{P}^1$ is proper, $JL(\pi \hat{\otimes} \hat{\mathcal{O}})$ is la adm for π la adm.