Shimura varieties and modularity (3/3)

Ana Caraiani — July 27

<u>Outline</u>
(1) Vanishing conjs & results for Shimura varieties
(2) The geometry of the Hodge-Tate period morphism
(3) Complementary applications
  ⇝ Caraiani - Scholze
  ⇝ Kashikawa
(4) Applications (via local-global) compatibility

Def Let k = finite field of char ℓ.

<u>Setups</u> (1) Let $p \neq \ell$ prime, $L/\mathbb{Q}_p$ finite ext'n

$$\bar{\rho} : \Gamma_L = \mathrm{Gal}(\bar{L}/L) \longrightarrow GL_n(k)$$

a conti repr.

Def'n We say that $\bar{\rho}$ is <u>generic</u> if
- it's unramified
- the eigenvalues of $\bar{\rho}(\mathrm{Frob}_L)$ $\{\lambda_1, \dots, \lambda_n\}$ satisfy

$$\lambda_i / \lambda_j \neq (\# k_L)^{\pm 1}, \quad i \neq j.$$

(2) Let F = number field &

$$\bar{\rho} : \Gamma_F = \mathrm{Gal}(\bar{F}/F) \longrightarrow GL_n(k)$$

a conti repr.

Def'n We say that a prime $p \neq \ell$ is <u>decomposed generic</u> for $\bar{\rho}$ if
- p splits completely in F
- $\forall v | p$ prime of F, $\bar{\rho}|_{\Gamma_{F_v}}$ is generic.

We say that $\bar{\rho}$ is <u>decomposed generic</u> if
$\exists$ one (thus infinitely many) decomposed generic prime for $\bar{\rho}$.

<u>Rmks</u> (i) In (i), genericity guarantees that any lift of $\bar{\rho}$ to char 0 corresponds to irred / generic principal series rep'ns of $GL_n(L)$.
Another way to view this:

$\bar{\rho}$ cannot be L-parameter of non-quasi-split G_b for $b \in B(GL_n, L)$.

(ii) Decomposed generic related to asking $\bar{\rho}$ to have large image.
<u>E.g.</u> if F tot real, $n=2$, $\bar{\rho}$ odd,
then $\mathrm{Im}(\bar{\rho}) \subset GL_2(k)$ non-solvable
$\Rightarrow \bar{\rho}$ decomposed generic.

Let $(B, *, V, \langle \cdot, \cdot \rangle)$ be a PEL datum of type A.
$\rightsquigarrow (\boxed{G}, X)$ Shimura datum
unitary similitude group
B = central simple alg w centre CM field F
$\rightsquigarrow Sh_K / E$, $\quad K \subset G(\mathbb{A}^\infty)$ sufficiently small.

$$\begin{array}{l} \pi^S(K) \circlearrowright H^*(Sh_K, \mathbb{F}_\ell) \\ \quad \cup \\ \quad \mathfrak{m} \\ \quad \Big\downarrow \text{ by Scholze (as in S.W. Shin's talk).} \\ \bar{\rho}_{\mathfrak{m}} : \Gamma_F \longrightarrow GL_n(\bar{\mathbb{F}}_\ell) \end{array}$$

<u>Conj III'</u> (Koshikawa) If $\bar{\rho}_{\mathfrak{m}}$ is decomposed generic, then

(i) $H^i_c(Sh_K, \mathbb{F}_\ell)_{\mathfrak{m}} \neq 0$

$\Rightarrow i \leq d = \dim_E Sh_K$

(ii) $H^i(Sh_K, \mathbb{F}_\ell)_{\mathfrak{m}} \neq 0$

$\Rightarrow i \geq d = \dim_E Sh_K$

Poincaré duality

In particular, if either Sh_K is compact or $\mathfrak{m}$ is non-Eisenstein then $H^*_c(Sh_K, \mathbb{Z}_\ell)_{\mathfrak{m}} \simeq H^*(Sh_K, \mathbb{Z}_\ell)_{\mathfrak{m}}$ is concentrated in degree d & torsion-free.

<u>Rmks</u> (i) If we fix a prime p that splits completely in F & s.t. $K_v = GL_n(\mathcal{O}_{F,v})$, $\forall v | p$ prime of F, then can formulate version of Conj III' only using spherical Hecke algebras at $v|p$ & their systems of eigenvalues.

(ii) Previous results towards Conj III' of

Lan-Suh $\}$ integral p-adic

Emerton-Gee $\}$ Hodge theory

Shin, — supercuspidal cond.

Boyer — Harris-Taylor case, Conj III.

<u>Thm</u> (Caraiani-Scholze, Koshikawa)

If Sh_K is compact, or $B = F$, $V = F^{2m}$ & G quasi-split, then the conjecture is true.

Rmk (i) Both approaches use geometry of Hodge-Tate morphism.

(ii) M. Santos is working on full Conj III

(where PEL type A, p splits completely in F).

via Caraiani-Scholze semi-perversity

+ Koshikawa approach.

§2 The geometry of the Hodge-Tate period morphism

$$\begin{tikzcd}
& \mathcal{S}^{tor}_{K^p} \arrow[d] \arrow[dd, bend left, "\pi^{tor}_{HT}"] \\
\mathcal{S}_{K^p} \arrow[ur] \arrow[r] \arrow[dr, "\pi_{HT}"'] & \mathcal{S}^{min}_{K^p} \arrow[d, "\pi^{min}_{HT}"] \\
& \mathcal{F}\ell_{G,\mu}
\end{tikzcd}$$

toroidal cpt

minimal (BB) cpt

$\mathcal{F}\ell_{G,\mu} = \mathcal{F}\ell$

For $? \in \{\phi, \text{tor}, \text{min}\}$, as diamonds,

$$\mathcal{S}^{?}_{K^p} = \varprojlim_{K_p} \mathcal{S}^{?}_{K^pK_p}.$$

(1) $\exists$ Newton stratification

$$\mathcal{F}\ell = \coprod_{b \in B(G,\mu)} \boxed{\mathcal{F}\ell^b}$$ ← loc closed strata

(Viehmann) $$\overline{\mathcal{F}\ell^b} = \coprod_{b' \geq b} \mathcal{F}\ell^{b'}$$ accounting Bruhat order.

· $\mathcal{F}\ell^{ord} = \mathcal{F}\ell(\mathbb{Q}_p)$. $\mathcal{F}\ell^{basic}$ open.

$$\begin{tikzcd}
\mathcal{F}\ell \simeq \mathrm{Gr}^{B^+_{dR}}_{G,\mu} \arrow[r] \arrow[dr] & \mathrm{Gr}^{B^+_{dR}}_{G} \arrow[d] \\
& \mathrm{Bun}_G
\end{tikzcd}$$

Newton stratification is

- pulled back from Bun_G.
- compatible under π_{HT} on rk 1 pts w/ Newton stratification on $\mathcal{F}\ell$.

(2) Igusa varieties

$\bar{S}_{K^pK_p} / \bar{\mathbb{F}}_p$, K_p hyperspecial.

$b \in B(G,\mu) \rightsquigarrow$ can find p-div gp w/ G-structure $\mathbb{X}_b / \bar{\mathbb{F}}_p$ s.t.

- isocrystal w/ G-str is b
- compatible w/ μ.

Oort central leaf

$$\mathscr{C}_{\mathbb{X}_b} = \left\{ x \in \bar{S}^b_{K^pK_p} \;\middle|\; \begin{array}{l} \exists \text{ isom } A[p^\infty] \times_{\bar{S}^b_{K^pK_p}} \kappa(\bar{x}) \simeq \mathbb{X}_b \times_{\bar{\mathbb{F}}_p} \kappa(\bar{x}) \\ \text{compatible w/ } G\text{-structures} \end{array} \right\}$$

$$\begin{array}{l} \mathrm{Ig}^b / \bar{\mathbb{F}}_p \text{ perfect scheme} \\ \quad \big\downarrow \text{ profinite} \\ \mathscr{C}_{\mathbb{X}_b} \end{array}$$

universal object which trivializes $A[p^\infty]|_{\mathscr{C}_{\mathbb{X}_b}}$.

$\exists$ isom $\quad A[p^\infty] \times_{\mathscr{C}_{\mathbb{X}_b}} \mathrm{Ig}^b \simeq \mathbb{X}_b \times_{\bar{\mathbb{F}}_p} \mathrm{Ig}^b$

compatible w/ G-str

$\mathrm{Ig}^{b,\text{pre-perf}} / \mathrm{Spa}(\breve{\mathbb{Q}}_p, \breve{\mathbb{Z}}_p)$.

(3) Rapoport-Zink spaces

$\mathcal{M}^b / \mathrm{Spf}\, \breve{\mathbb{Z}}_p$ formal sch w/ moduli theoretic description

$R = \breve{\mathbb{Z}}_p$-alg on which p nilpotent

$$R \longrightarrow \mathfrak{M}^b(R) = \left\{ (\mathcal{G}, \rho) : \begin{array}{l} \mathcal{G} \text{ p-div gp w/ } G\text{-str} / R \\ \rho : \mathbb{X}_b \times_{\overline{\mathbb{F}}_p} R/p \longrightarrow \mathcal{G} \times_R R/p \text{ quasi-isog} \end{array} \right\}$$

Set $\underbrace{\mathcal{M}^b = (\mathfrak{M}^b)^{ad}_\eta}_{\text{adic space assoc to } \mathfrak{M}^b.}$

$\rightsquigarrow$ $\mathcal{M}^b_\infty \xrightarrow{\pi_{HT}} \mathcal{F}\ell^b$ pre-perfectoid over $/ \mathbb{Q}_p(\zeta_{p^\infty})$

(4) Product formula

$\exists$ cartesian diagram of diamonds ($\forall b \in B(G, \mu)$)

$$\begin{array}{ccc} \mathcal{M}^b_\infty \times_{\mathrm{Spa}(\breve{\mathbb{Q}}_p, \breve{\mathbb{Z}}_p)} \mathrm{Ig}^{b, \text{pre-perf}} & \longrightarrow & \mathcal{M}^b_\infty \\ \downarrow & \square & \downarrow \pi^b_{HT}. \\ \underbrace{(\mathcal{Y}^\circ_{K^p})^b}_{\text{"good reduction locus"}} & \xrightarrow{\pi_{HT}} & \mathcal{F}\ell^b \end{array}$$

pro-étale torsor for $\tilde{G}_b$.

- infinite-level version of Mantovian product formula
- Cohomological consequence

$$R\Gamma_c(Sh_{K^p}, \overline{\mathbb{F}}_\ell)$$

has a "filtration" by

$$\underset{\circlearrowleft \, G_b(\mathbb{Q}_p)}{R\Gamma(\mathrm{Ig}^b, \overline{\mathbb{F}}_\ell)^{op}} \overset{L}{\otimes}_{C_c(G_b(\mathbb{Q}_p))} R\Gamma_c(\mathcal{M}_\infty, \overline{\mathbb{F}}_\ell(d_b))[2d_b]$$

(i) $R\Gamma(\mathrm{Ig}^b, \overline{\mathbb{F}}_\ell)^{op}_{\mathfrak{m}} = 0$ by Caraiani-Scholze

$\Leftarrow$ $\mathfrak{m}$ generic + G_b not quasi-split.

Shin: computed $[H(\mathrm{Ig}^b, \overline{\mathbb{Q}}_\ell)_{\mathfrak{m}}]$

(ii) $R\Gamma_c(\mathcal{M}_\infty, \overline{\mathbb{F}}_\ell(d_b))_{\mathfrak{m}}[2d_b] = 0$ by Kashikawa.

$L / \mathbb{Q}_p$ fin ext'n, (G, b, μ) local Shimura datum

s.t. $G = \prod_{i \in I} GL_{n_i} \; / L$

$K = \prod_{i \in I} GL_{n_i}(\mathcal{O}_L) \subset G(L)$.

$\rightsquigarrow \mathcal{M}_K$ RZ space / local Shimura variety.

Thm Assume $\ell \neq p$. If $\mathfrak{m}' \subset \mathcal{H}_K \circlearrowright H^i_c(\mathcal{M}_K, \mathbb{Z}_\ell)$,

s.t. $\rho_{\mathfrak{m}'}$ is generic & G_b non-quasi-split,

then $\forall i$, $H^i_c(\mathcal{M}_K, \mathbb{Z}_\ell)_{\mathfrak{m}} = 0$.

Pf idea

$$\begin{array}{c} H^i_c(\mathcal{M}_K, \overline{\mathbb{F}}_\ell) \\ \circlearrowleft \quad \circlearrowright \\ \mathcal{H}_K \quad G_b(L) \end{array}$$

- π sm irrep / $\overline{\mathbb{F}}_\ell$ of $G_b(L)$

  $\rightsquigarrow \varphi_\pi$ not generic whenever $G_b(L)$ not quasi-split.

  L-parameter constr'd by Fargue-Scholze.

- π irr subquotient of $H^i_c(\mathcal{M}_K, \overline{\mathbb{F}}_\ell)_{\mathfrak{m}}$

  $\rightsquigarrow \varphi_\pi = \rho_{\mathfrak{m}}$

  Fargues-Scholze excursion operators.