Explicit construction of automorphic forms (1/2)

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Lecture 1 Poincaré series.

Θ-correspondence — traditional Autom
— Interaction with LP
— Interaction with RLP.

Lecture 2 Doubling & descent.

§1 Poincaré series

Simple question How do you know $\exists$ nonzero cusp forms?

Poincaré series: G split ss $/\mathbb{Q}$.

$$P : C_c^\infty(G(\mathbb{A})) \longrightarrow C_c^\infty([G]) \qquad G\text{-equivariant.}$$

$$f \longmapsto P(f)(g) := \sum_{\gamma \in G(\mathbb{Q})} \underbrace{f(\gamma g)}_{\text{left } G(\mathbb{Q})\text{-invariant}}$$

a finite sum since $|G(\mathbb{Q}) \cap \mathrm{Supp}(f) g^{-1}| < \infty$.

Q Which $f = \prod_v f_v$?

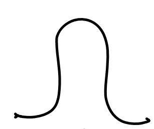
$\rightsquigarrow$ Fix p & sc rep π of $G(\mathbb{Q}_p)$. Then

$$f_v = \begin{cases} \langle \cdot v, v \rangle_\pi, & v = p, \\ 1_{G(\mathbb{Z}_p)}, & v \neq p \text{ finite} \\ f_\infty & v = \infty \end{cases}$$

to be chosen later.

By making $\mathrm{Supp}(f_\infty)$ suff small,

$$|G(\mathbb{Q}) \cap \mathrm{Supp}(f)| = \{1_G\}$$



Supported around 1_G

$\rightsquigarrow P(f)(1) = f(1) \neq 0$

$\rightsquigarrow\ 0 \neq P(f) \in L^2([G])$.

Note π sc $\Rightarrow$ all const terms of f vanish.

$\Rightarrow 0 \neq P(f) \in L^2_{cusp}([G])$.

"globalization of π".

Take $\Pi = \otimes'_v \Pi_v$ = irred summand of $\langle G(\mathbb{A}) \cdot P(f) \rangle$ satisfies

- $\Pi_p \simeq \pi$,
- Π_v unram, $\forall v \neq p$ finite
- Π_∞ no info provided.

Now varying $p \rightsquigarrow$ get infinitely many cuspidal rep's.

Q How do you know $\mathrm{Irr}_{sc}\, G(\mathbb{Q}_p) \neq \emptyset$?

Ans Take a cuspidal rep τ of $G(\mathbb{F}_p)$ (automatically lifts to $G(\mathbb{Z}_p)$).

& Set $\boxed{\pi = c\text{-}\mathrm{Ind}_{G(\mathbb{Z}_p)}^{G(\mathbb{Q}_p)} \tau}$ (irred sc, depth 0).

Q How do you know $\mathrm{Irr}_{cusp}\, G(\mathbb{F}_p)$?

Ans Deligne - Lusztig $R_{T,\theta}$.

Rmks (i) Variants: If $H \subset G$ reductive, π H-dist,

$\rightsquigarrow \exists$ globalization Π of π s.t. Π H-dist as well.

(Prasad - Scholze - Pillot).

(ii) Does not work for D.S. π

Poincaré series + inputs from fell top / weak containment.

$\Rightarrow$ globalization [SV].

(iii) For $K_f(x,y) = \sum_{\gamma \in G(\mathbb{Q})} f(x^{-1}\gamma y)$:

$$K_f(x,y) \begin{cases} \xrightarrow{\int_{[G^\Delta]}} TF \\ \xrightarrow{\int_{[H_1]}\int_{[H_2]}} RTF. \end{cases}$$

$P_f(y) = K_f(1,y)$.

§2 Θ-Correspondence

<u>Howe-PS</u> (Corvallis) Using Θ-corr, they constructed for E/F quad ext'n that

$$\Theta : \left\{ \begin{array}{l} \text{Hecke characters of} \\ E^\times/F^\times, \text{ not factoring} \\ \text{through } N_{E/F} \end{array} \right\} \longrightarrow \left\{ \begin{array}{c} \text{Cuspidal rep'ns} \\ \text{of } GSp_4 \end{array} \right\}$$

non-tempered!

General framework (local)

$G \times H \xrightarrow{\iota} \mathcal{E}$. Ω rep'n of $\mathcal{E}$.

Spectral decomp of $\iota^\# \Omega$

$\rightsquigarrow$ correspondence $\Sigma \subseteq \mathrm{Irr}\, G \times \mathrm{Irr}\, H$

$\Sigma = \{(\pi, \sigma) : \Omega \twoheadrightarrow \pi \otimes \sigma\}$.

<u>An ideal situation</u> Σ is the graph of a map

$\Theta_\Omega : \mathrm{Irr}\, G \longrightarrow \mathrm{Irr}\, H$.

(Global ver) $\phi \in \boxed{\Omega_{/\mathbb{A}}} = \bigotimes_v{}' \Omega_v \xrightarrow{\Theta} \mathcal{A}([\mathcal{E}])$, $\circlearrowleft \mathcal{E}_{/\mathbb{A}}$

$$\mathcal{A}([\mathcal{E}]) \xrightarrow{\text{rest}} C([G \times H])$$

$\Theta(\phi)$ gives $\mathcal{A}(G) \longrightarrow \mathcal{A}(H)$.

<u>Main players</u>
- W symplectic v.s.
- V quadratic v.s.

$\Rightarrow \underset{G}{Sp(W)} \times \underset{H}{O(V)} \longrightarrow \underset{\mathcal{E}}{Sp(W \otimes V)}$

E.g. $O_2 \times Sp_4$, $SO_2 \simeq E^\times / F^\times$.

$\psi : F \to \mathbb{C}^\times$, $\Omega = \Omega_\psi$.

Can determine Weil rep'n of

$$\begin{tikzcd}
\text{metaplectic gp} \to Mp(V \otimes W) \arrow[d, "2"] & & \\
Sp(V \otimes W) & & Sp(W) \times O(V) \arrow[ll, "2"'] \arrow[ull, dashed, "\tilde{\iota}"']
\end{tikzcd}$$

analyzed by Kudla.

called an oscillator

Schödinge model (relation w/ quantum mechanics)

Ω realized on $W = X \oplus Y$ with decomp

$\rightsquigarrow$ $\Omega = S(V \otimes Y)$ source functions

e.g. $h \in O(V)$, $(h \cdot \phi)(y) = \phi(h^{-1} y)$

Main questions $G = Sp(W)$, $H = O(V)$.

(a) (Local smooth) For $\sigma \in \mathrm{Irr}\, G$,

define multiplicity space

$$\Theta(\sigma) := (\Omega \otimes \sigma^\vee)_G . \quad \circlearrowleft H$$

asking whether σ appears as a quotient or not.

Called big Θ-lift of σ.

Note $\Omega \otimes \sigma^\vee \twoheadrightarrow \Theta(\sigma)$

$\rightsquigarrow$ $\Omega \twoheadrightarrow \sigma \otimes \Theta(\sigma)$ (max'l σ-isotypic quotient.)

Thm (Howe duality)

$\Theta(\sigma)$ has finite length (as H-mod)

& a unique irred quotient (if nonzero)

denoted by $\theta(\sigma)$ (small θ-lift).

$\Rightarrow \Theta : \mathrm{Irr}(G) \longrightarrow \mathrm{Irr}(H) \cup \{0\}$.

<u>Rmk</u> If $G \ll H$, $\Theta : \mathrm{Irr}\, G \longrightarrow \mathrm{Irr}(H)$ (can suppress $\{0\}$.)

<u>Thm</u> (Cont) If $\sigma \neq \sigma'$, then $\Theta(\sigma) \neq \Theta(\sigma')$ if they are nonzero.

Namely, "$\Theta : \mathrm{Irr}\, G \hookrightarrow \mathrm{Irr}\, H$" is injective if $G \ll H$

outside the fiber of 0.

(b) (Local L^2) $\Omega = S(V \otimes Y) \hookrightarrow \hat{\Omega} = L^2(V \otimes Y)$.

<u>Thm</u> (Sakellaridis) Assume $G < H$. Then

$$\hat{\Omega} = \int_{\hat{G}} \sigma \otimes \Theta_{L^2}(\sigma)\, \underbrace{d\mu_G(\sigma)}_{\text{H-C Plancherel measure}}.$$

(so $\mathrm{supp}(\hat{\Omega}) \subseteq \hat{G}_{temp}$.) $\quad \Theta^{\infty}_{L^2}(\sigma) = 0$ or $\Theta(\sigma)$

Precisely, for $\phi_1, \phi_2 \in \Omega$,

$$\langle \phi_1, \phi_2 \rangle_{\hat{\Omega}} = \int_{\hat{G}} J_\sigma(\phi_1, \phi_2)\, d\mu_G(\sigma)$$

with $J_\sigma(\phi_1, \phi_2) = \sum_{f \in ONB(\sigma)} Z_\sigma(\phi_1, \phi_2, f, f)$ through orthonormal basis

$$Z_\sigma(\phi_1, \phi_2, f, f) = \int_G \langle g\phi_1, \phi_2 \rangle \overline{\langle gf, f \rangle}\, dg$$

↑ local doubling theta integral.

<u>Rmk</u> Indeed, Z_σ first appears in J.S. Li's thesis work.

$$\begin{tikzcd} Z_\sigma : \Omega \otimes \bar{\Omega} \otimes \bar{\sigma} \otimes \sigma \arrow[rr, "G^\Delta \times H^\Delta\text{-inv}"] \arrow[dr] & & \mathbb{C} \\ & \Theta(\sigma) \otimes \overline{\Theta(\sigma)} \arrow[ur] & \end{tikzcd}$$

<u>Lemma</u> $Z_\sigma \neq 0 \iff \Theta(\sigma^\infty) \neq 0$

explicit side $\quad$ abstract side

(ok if F non-arch, but not known for $\mathbb{C}$.)

J.S.-Li: If $G \ll H$ (stable range),

Z_σ descends to a positive inner product on $\Theta(\sigma)$ if σ unitary.

<u>Global setting</u>

$$\Omega_{\mathbb{A}} = S(Y_{\mathbb{A}} \otimes V_{\mathbb{A}}) \xrightarrow{\theta} C([G \times H])$$

$$\phi \longmapsto \theta(\phi)(g,h) = \sum_{y \in Y_F \otimes V_F} ((g,h)\cdot\phi)(y)$$

(like spherical varieties)

theta series

For $\phi \in \Omega_{\mathbb{A}}$, $f \in \sigma \subseteq \mathcal{A}_{cusp}[G]$, set

$$\theta(\phi, f)(h) = \int_{[G]} \theta(\phi)(g,h)\, \overline{f(g)}\, dg.$$

$\rightsquigarrow$ $\Theta(\sigma) := \langle \theta(\phi, f),\ \phi \in \Omega_{\mathbb{A}}, f \in \sigma \rangle \subseteq \mathcal{A}[H]$.

<u>Questions</u>

- Is $\Theta(\sigma) \neq 0$?
- Is $\Theta(\sigma) \in \mathcal{A}_{cusp}[H]$? or $\mathcal{A}_{L^2}[H]$?
- What is it?

<u>Prop</u> If $0 \neq \Theta(\sigma) \subseteq \mathcal{A}_{L^2}(H)$, then $\Theta(\sigma)$ is irred and $\Theta(\sigma) \cong \bigotimes_v' \theta(\sigma_v)$.

<u>Non-vanishing</u> Is $\theta(\phi, f) \neq 0$?

Consider $\langle \theta(\phi_1, f_1), \theta(\phi_2, f_2) \rangle_{Pet}$ (Rallis inner product formula)

L^2-product / Petersson product

$$\| \qquad (*) \cdot \prod_v^* Z_{\sigma_v}(\phi_1, \phi_2, f_1, f_2),$$

Equivalently: $\langle \theta(\phi_1)_\sigma, \theta(\phi_2)_\sigma \rangle_{Pet} = (*) \prod_v^* J_{\sigma_v}(\phi_1, \phi_2)$.

Rmk $\prod_v^* Z_{\sigma_v} \sim L((\dim V - \dim W)/2, \sigma, \mathrm{Std})$
an autom L-function.

Summary · Global: L-functions
· Local: Is $\Theta(\sigma) \neq 0$? What is it?
What is Θ: "$\mathrm{Irr}\, G \to \mathrm{Irr}\, H$" locally?
(what's its fiber at 0?)

Answers (of questions locally about $\Theta(\sigma)$ above).

	Spherical $X \circlearrowleft G$	Θ-Correspondence
Local smooth	$C_c^\infty(X) \twoheadrightarrow \pi$	$\Omega \twoheadrightarrow \sigma \otimes \Theta(\sigma)$
Dual data	$X^\vee \times SL_2 \to G^\vee$	$Sp(W)^\vee \times SL_2 \to Sp(W)^\vee \times O(V)^\vee$
Local L^2	$L^2(X) = \int_{\hat{G}_X} \pi_\sigma \, d\mu_{G_X}(\sigma)$	$\hat{\Omega} = \int_{\widehat{Sp(W)}} \sigma \otimes \Theta(\sigma) \, d\mu_{Sp(W)}(\sigma)$
Local Inner Product	$\langle f_1, f_2 \rangle = \int_{\hat{G}_X} J_{\pi_\sigma}(f_1, f_2)$	$\langle \phi_1, \phi_2 \rangle = \int_{\widehat{Sp(W)}} J_\sigma(\phi_1, \phi_2)$
Global Inner Product	$\langle \Theta(f_1)_\pi, \Theta(f_2)_\pi \rangle$ $\sim \prod_v^* J_{\pi_v}(f_{1,v}, f_{2,v})$	$\langle \Theta(\phi_1)_\sigma, \Theta(\phi_2)_\sigma \rangle$ $= \prod_v^* J_{\sigma_v}(\phi_v, \phi_v)$
	$\langle \Theta f_1, \Theta f_2 \rangle = RTF(X \times X / G)$	$\langle \Theta(\phi_1), \Theta(\phi_2) \rangle = \text{Siegel-Weil}$
L-functions	$L_X(s) = L(s, V_X)$	$L(s+\frac{1}{2}, \mathrm{Std}) \cdot L(s+1, \mathrm{Ad})$

§3 Extended RLP

Q What G-mod should RLP be connected with?
Ben-Zvi, Sakellaridis, Venkatesh:
G-mods that arise as quantizations of
certain Hamiltonian G-vars.

Hamiltonian G-var M:

$M \circlearrowleft G$ symplectic var

$+ \ \mu : M \longrightarrow \mathfrak{g}^* = \mathrm{Lie}\, G$ moment map

classical mechanics

quantization

$G \circlearrowright \Pi_M$ unitary

quantum mechanics

Also take $\mathfrak{X} \subseteq M$ Lagrangian submanifold

$\rightsquigarrow \Pi_M = L^2(\mathfrak{X})$.

E.g. · (A simple example)

W symplectic v.s. w/ $W \circlearrowleft G = Sp(W)$ or $Mp(W)$.

$$\mu : W \longrightarrow \mathfrak{g}^*$$

$$w \longmapsto (X \longmapsto \langle Xw, w \rangle).$$

$\rightsquigarrow W = X \oplus Y$, $\Pi_W = L^2(Y)$ Weil rep'n

$$\begin{array}{c} L^2(Y) \\ \circlearrowleft \\ Mp(W) \end{array}$$

· $M = T^* X$, $X \circlearrowright G$

$$\begin{array}{c} M = T^*X \\ \uparrow \text{zero section (as Lagrangian)} \\ X \end{array}$$

$X \rightsquigarrow \Pi_M = L^2(X) \circlearrowright G$.

· $\mathfrak{g}^*/G \xrightarrow{\text{orbit manifold}} \widehat{G}$.