Explicit construction of automorphic forms (2/2)

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<u>Lecture 1</u> Poincaré series & Θ-correspondence

<u>Lecture 2</u> Doubling & descent $\left\{\begin{array}{l}\text{The doubling method}\\ \text{Generalized doubling}\\ \text{Double descent}\\ \text{The original descent}\end{array}\right.$

<u>Recall Θ-corr</u>:

Global Θ-lift $\langle \theta(\phi, f_1), \theta(\phi, f_2) \rangle$

$$\sim (*) \cdot \prod_v^{*} \underbrace{Z_v(\phi_{1,v}, \phi_{2,v}, f_{1,v}, f_{2,v})}_{\text{called doubling local zeta integral}}$$

Rankin-Selberg integral for standard L-fcn for $G \times GL_1$ (G classical)

<u>See-saw dual pairs</u>

$\mathcal{E}$ ambient grp $\rightsquigarrow$

$$\begin{tikzcd} & \mathcal{E} & \\ G \times H \arrow[ur] & & G' \times H' \arrow[ul] \end{tikzcd} \qquad \Omega \text{ Weil rep}$$

2 dual pairs, $G \subset G'$, $H \supset H'$, with

$$\begin{tikzcd} G' \arrow[dr, dash] & & H \arrow[dl, dash] \\ G \arrow[u, hook] \arrow[rr, leftrightarrow, "\Theta"] & & H' \arrow[u, hook] \end{tikzcd}$$

$\mathrm{Irr}\, G \ni \pi \qquad \sigma' \in \mathrm{Irr}\, H$

(formulated by Kudla in 1980s)

See-saw identity: $\mathrm{Hom}_{G\times H'}(\Omega, \pi\boxtimes\sigma')$

$$\mathrm{Hom}_G(\Theta(\sigma'),\pi) = \mathrm{Hom}_{G\times H'}(\Omega, \pi\boxtimes\sigma') = \mathrm{Hom}_{H'}(\Theta(\pi),\sigma')$$

useful triviality

Doubling See-saw $\quad Sp(W)\times O(V) \quad (\dim W \underset{\text{Can remove this}}{<} \dim V)$

Set $W^\square = W\oplus W^- \overset{\Delta}{\hookleftarrow} W^\Delta$ max isotropic (b/c half-dim)

negate the symplectic form: (W,ω) & $(W^-,-\omega)$.

Inside

$$\begin{tikzcd}[column sep=small] & Sp(V\otimes W^\square) & \\ O(V)\times Sp(W^\square) \arrow[ur] & & O(V)^2\times Sp(W)^2 \arrow[ul] \end{tikzcd}$$

have 2 dual pairs

Have the doubling see-saw:

$$\begin{tikzcd} Sp(W^\square) \arrow[drr, dash] & & O(V)\times O(V) \\ \pi'\boxtimes\pi^\vee \quad Sp(W)\times Sp(W^-) \arrow[u, hook] \arrow[urr, dash] \arrow[rr, leftrightarrow, "\Theta"] & & O(V) \quad 1 \arrow[u, hook, "\Delta"'] \end{tikzcd}$$

$\rightsquigarrow$ See-saw identity:

$$\mathrm{Hom}_{Sp(W)\times Sp(W^-)}(\Theta(1), \pi'\boxtimes\pi^\vee)$$
$$\cong \mathrm{Hom}_{O(V)^\Delta}(\Theta(\pi')\boxtimes\Theta(\pi), \mathbb{C})$$
$$\cup\!\!\!\mid$$
$$\mathrm{Hom}_{O(V)^\Delta}(\underbrace{\theta(\pi')\boxtimes\theta(\pi)}, \mathbb{C})$$

$\theta(\cdot)$ is the quotient of $\Theta(\cdot)$

& is either 0 or irred. $\Leftarrow$ HD.

$$= \mathrm{Hom}_{O(V)}(\theta(\pi'),\theta(\pi)).$$

Howe duality (inequality formulation)

$$\dim \mathrm{Hom}_{O(V)}(\theta(\pi'),\theta(\pi)) \le \delta_{\pi,\pi'} \le 1.$$

Thm of Rallis $\Theta(1) \hookrightarrow I_{P(W^\Delta)}^{Sp(W^\square)} |\det|^{s_0}$ $= GL(W^\Delta)\cdot N(W^\Delta)$

rep of $Sp(W^\square)$ W^Δ Siegel parabolic

$\rightsquigarrow$ $\mathrm{Hom}_{Sp(W)\times Sp(W^-)}(\Theta(1), \pi' \boxtimes \pi^\vee)$

$\approx \mathrm{Hom}_{Sp(W)\times Sp(W^-)}(I_{P(W^\Delta)}^{Sp(W^\square)}(s_0), \pi' \boxtimes \pi^\vee)$.

Lemma As $Sp(W)\times Sp(W^-)$-module,

$$I_{P(W^\Delta)}(s) \longleftarrow C_c^\infty(Sp(W))$$

with small cokernel.

$[W^\Delta] \in P(W^\Delta)\backslash Sp(W^\square) \curvearrowleft Sp(W)\times Sp(W^-)$

w/ open dense orbit

& stabilizer $\mathrm{Stab}(W^\Delta) = Sp(W)^\Delta$.

$\rightsquigarrow$ $\mathrm{Hom}_{Sp(W)\times Sp(W^-)}(I_{P(W^\Delta)}^{Sp(W^\square)}(s_0), \pi' \boxtimes \pi^\vee)$.

$\approx \mathrm{Hom}_{Sp(W)\times Sp(W^-)}(C_c^\infty(Sp(W)), \pi' \boxtimes \pi^\vee)$ & has $\dim = \delta_{\pi',\pi}$.

Global $\langle \Theta(\phi_1, f_1), \Theta(\phi_2, f_2)\rangle_{O(V)}$, $f_1 \in \pi_1$, $f_2 \in \pi_2$

$= \int_{[O(V)]} \Theta(\phi_1, f_1)(h)\, \overline{\Theta(\phi_2, f_2)(h)}\, dh$

$= \int_{[O(V)]} \left(\int_{[Sp(W)]} \Theta(\phi_1)(g_1, h) \overline{f_1(h)} \int_{[Sp(W^-)]} \Theta(\phi_2)(g_2, h) \overline{f_2(h)}\, dg_1 dg_2 \right) dh$

$= \int_{[Sp(W)\times Sp(W^-)]} \overline{f_1(g)} f_2(g)$ $\quad dh \Big) dg_1 dg_2$

(Siegel-Weil Thm (Roughly, Kudla-Rallis)

$\langle \Theta(\phi_1), \Theta(\phi_2)\rangle = \mathcal{E}(s_0, \Phi_{\phi_1 \otimes \overline{\phi_2}})$

Θ-"lift of $1_{O(V)}$ to $Sp(W^\square)$)

Back to computation:

global doubling zeta integral.

$\langle \Theta(\phi_1, f_1), \Theta(\phi_2, f_2)\rangle_{O(V)^\Delta} = \int_{[Sp(W)^2]} \overline{f_1(g_1)} f_2(g_2) \cdot \mathcal{E}(s_0, \Phi_{\phi_1 \otimes \overline{\phi_2}}, (g_1, g_2))\, dg_1\, dg_2$

Consider $I_P(s,\chi) = \mathrm{Ind}_{P(W^\square)}^{Sp(W^\square)} \chi \cdot |\det|^s$

$\rightsquigarrow \mathcal{E}(s,\chi)$ Eisenstein Series

$$Z(s,f_1,f_2,\Phi) = \int_{[Sp(W)]^2} f_1(g_1) f_2(g_2)\, \mathcal{E}(s,\chi,(g_1,g_2))\, dg_1\, dg_2$$

Inputs: $f_1 \in \pi_1$, $f_2 \in \pi_2$, $\Phi_s \in I_P(s,\chi)$.

$\rightsquigarrow$ This
- converges at all s, where $\mathcal{E}(s,\chi)$ is hol.
- vanishes unless $\pi_1 \simeq \pi_2$
- if $\pi_1 = \pi_2 = \pi$, then $Z(s,f_1,f_2,\chi) \approx L(s+\frac{1}{2}, \pi\times\chi, \mathrm{std})$.

Generalized doubling

(Cai - Friedberg - Ginzburg - Kaplan)

$\rightsquigarrow$ get Rankin-Selberg integral rep for $L(s,\pi\times\tau,\mathrm{std})$

w/ π cusp of $Sp(2n)$,

τ cusp of $GL(k)$, $\forall \boxed{k\geq 1}$

Application (Cai - Friedberg - Kaplan)

Combining this & Converse thm of Cogdell-PS, show weak lifting (classical grps) $\to$ (GL).

dim $W = 2n$,
dim $W^\square = 4n$

Levi $= GL(2nk)$

$$\begin{pmatrix} \square_k & & & \\ & \square & & \\ & & \ddots & \\ & & & \square \end{pmatrix}$$

2n blocks

- $\mathrm{Ind}_P^{\boxed{Sp(4nk)}} \Delta(\tau,2n)\cdot|\det|^s = I_P(\Delta(\tau,2n),s)$

$\Delta(\tau,2n) = $ Speh repr of $GL(2n,k)$

$$= LQ\left(\tau\cdot|\det|^{\frac{2n-1}{2}} \times \tau\,|\det|^{\frac{2n-3}{2}} \times\cdots\times \tau\,|\det|^{-\frac{2n-1}{2}}\right)$$

a unique irred quotient called Langlands quotient

$\hookrightarrow \mathcal{A}_{res}(GL(2nk))$ (Mœglin - Waldspurger)

Let $\mathcal{E}(s,\Delta(\tau,2n),\Phi)$ be assoc Eis series

w/ $f_1\in\pi_1$, $f_2\in\pi_2$, $\Phi\in I_P(s,\Delta(\tau,2n))$

$\rightsquigarrow \quad Z(s, f_1, f_2, \Phi) \approx L(s+\tfrac{1}{2}, \pi\times\tau, \mathrm{Std}\otimes\mathrm{Std})$

$$= \int_{[Sp(2n)^2]} \overline{f_1(g_1)} \cdot f_2(g_2) \cdot \mathcal{E}^{(U,\psi_U)}(s, \Delta(\tau,2n), \Phi)(g_1,g_2)\, dg_1\, dg_2$$

Here $G \supset U \rtimes H$, $\psi_U : U \to \mathbb{C}^\times$ fixed by H.

Think locally: $\underset{\mathrm{Rep}\,G}{\pi \,\in} \rightsquigarrow \underset{\mathrm{Rep}\,H}{\pi_{U,\psi_U} \,\in} \circlearrowright H$. $\quad \mathcal{C}([H])$

via $\mathcal{E} \longmapsto \mathcal{E}^{U,\psi_U} \in \mathcal{C}([H])$ taking Fourier coeff by

$$\mathcal{E}^{U,\psi_U}(h) = \int_{[U]} \overline{\psi_U(u)} \cdot \mathcal{E}(uh)\, du$$

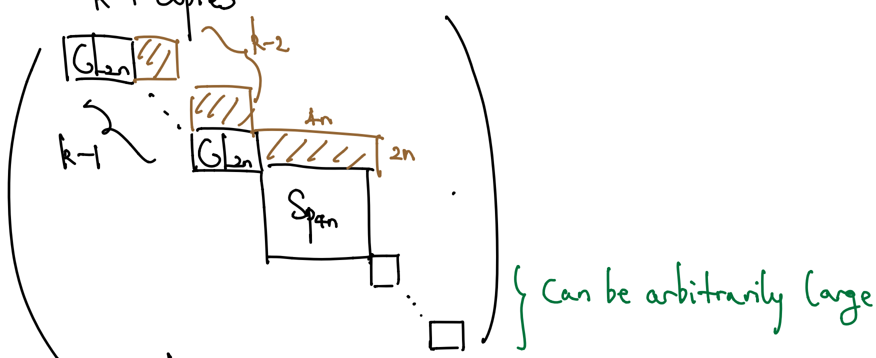
$[U]$ compact open.

E.g. In $Sp(4nk)$, there is a $U \rtimes (Sp_{2n} \times Sp_{2n})$

$\psi_U : U \to \mathbb{C}$ fixed by Sp_{2n}^2.

U = unipotent radical of parabolic Q with Levi

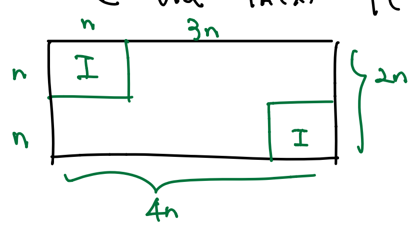
$\rightsquigarrow \quad U = \underbrace{GL_{2n} \times \cdots \times GL_{2n}}_{k-1 \text{ copies}} \times Sp(4n)$ in this case



$\rightsquigarrow \quad U/[U,U] \simeq \prod_{i=1}^{k-2} M_{2n\times 2n} \times M_{2n\times 4n}$

"most nondegenerate characters".

& $\psi_A : M_{n\times n} \longrightarrow \mathbb{C}$ via $\psi_A(x) = \psi(\mathrm{Tr}\, AX)$



$$\phi \in \mathrm{Hom}(\underset{\substack{\uparrow \\ \dim 4n}}{W_{4n}}, \underset{\substack{\uparrow \\ \dim 2n}}{V_{2n}}) \curvearrowleft Sp_{4n} \times GL_{2n} \qquad \text{s.t. } \phi|_{W_{2n}^-} = 0.$$

$$\& \ \phi|_{W_{2n}^+} : W_{2n}^+ \xrightarrow{\sim} V_{2n}.$$

$W_{4n} = W_{2n}^+ \oplus W_{2n}^-$ polarization

$$\begin{pmatrix} & & \\ & Sp(W_{2n}^-) & \\ & & \end{pmatrix}$$

$$\mathrm{Stab}_{Sp_{4n}\times GL_{2n}}(\phi) \cong Sp_{2n} \times Sp_{2n} \hookrightarrow Sp_{4n} \times GL_{2n}.$$

$$(A, B) \longmapsto ((A, B), \ B)$$

Application: Double descent

<u>Descent</u> $Sp(2n) \longrightarrow GL(2n+1)$

$$\begin{tikzcd} \pi \arrow[r] & \Pi \arrow[l, bend left, "\text{descent}"] \end{tikzcd} : \ L(1, \Pi, \mathrm{Sym}^2) = \infty.$$

$$\Rightarrow L^S(s, \pi \times \Pi^\vee) = L^S(s, \Pi \times \Pi^\vee) \text{ has a pole at } s=1$$

Given $\displaystyle\int \overline{f_1(g_1)}\, f_2(g_2)\, \mathcal{E}^{U,\psi_U}(g_1, g_2)\, dg_1\, dg_2 \approx L(s+\tfrac{1}{2}, \pi \times \tau)$

Take $\tau = \Pi$

$\rightsquigarrow \displaystyle\int \overline{f_1(g_1)}\, f_2(g_2) \cdot \mathrm{Res}_{s=1}(\mathcal{E}^{U,\psi_U}(g_1, g_2))\, dg_1\, dg_2$

$$\approx \mathrm{Res}_{s=\frac{1}{2}} L(s+\tfrac{1}{2}, \pi \times \Pi)$$

<u>Def</u> (Ginzberg-Soudry)

$$DD(\Pi) := \underset{\substack{\uparrow \\ \text{cuspidal proj}}}{P_{cusp}} \langle \mathrm{Res}_{s=\frac{1}{2}} \mathcal{E}^{U,\psi_U}(\bar{\Phi}) \rangle \subseteq L^2_{cusp}([Sp_{2n} \times Sp_{2n}])$$

Thm $\quad DD(\Pi) \simeq \bigoplus_{\pi \in \Sigma_\Pi} \pi \otimes \pi^\vee$

where $\Sigma_\Pi = \{\pi : \pi \text{ weakly lifts to } \Pi\}$.